Physics 745 - Group Theory

Homework Set 32

Due Monday, April 27

1. The group SO(4) has six generators, which can be chosen to be

$$L_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$K_1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}.$$

These can be shown to satisfy the commutation relations

$$[L_a, L_b] = i\varepsilon_{abc} L_c, \quad [L_a, K_b] = i\varepsilon_{abc} K_c, \quad [K_a, K_b] = i\varepsilon_{abc} L_c.$$

(a) This group is rank two, so we can pick two of these matrices to be mutually commuting. If I pick $H_1 = L_3$, what should I pick for H_2?

(b) Now, combine the remaining four operators into pairs, which I call $L_\pm$ and $K_\pm$, having the property

$$[H_1, L_\pm] = \pm L_\pm \quad \text{and} \quad [H_1, K_\pm] = \pm K_\pm$$

I'm not going to tell you how to do this, you have to guess for yourself.

(c) Unfortunately the operators you found in part (b) probably do not have simple commutation relations with H_2. Combine $L_\pm$ with $K_\pm$ to make two new operators, which I called $E_\pm$ and $F_\pm$, such that the commutation relations will always be proportional, *i.e.*,

$$[H_1, E_\pm] \propto E_\pm, \quad [H_2, E_\pm] \propto E_\pm, \quad [H_1, F_\pm] \propto F_\pm, \quad [H_2, F_\pm] \propto F_\pm.$$

(d) What are the roots of this group? Make a root diagram. Don't forget the roots corresponding to H_1 and H_2!