Physics 745 - Group Theory

Solution Set 28

1. [15] The group SU(3) contains the group SU(2) as a subgroup, and in more than one way

(a) [7] Show that the generators T_1, T_2 and T_3 form an SU(2) subgroup; that is, show that $[T_1, T_2] = iT_3$, etc. To save time, only do two of the three commutators. How does the 3 representation of SU(3) break into representations under this subgroup?

We simply work out the commutators directly:

$$[T_1,T_2] = \frac{1}{4}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{pmatrix} = iT_3,$$

$$[T_2,T_3] = \frac{1}{4}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = iT_2,$$

$$[T_3,T_1] = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = iT_2.$$

That was boring. Now, technically, we don't actually have to *do* anything to demonstrate this, because the matrices are already block diagonal. But in general, we would find the eigenvalues of T_3, which can be read off directly from this diagonal matrix, and the eigenvalues are $\left\{\tfrac{1}{2}, -\tfrac{1}{2}, 0\right\}$. The highest weight is $\tfrac{1}{2}$, which tells us we have the $\left(\tfrac{1}{2}\right)$ representation, which accounts for the weights $\pm\tfrac{1}{2}$. This leaves the weight 0, which corresponds to the (0) representation, so

$$3 \to \left(\tfrac{1}{2}\right) \oplus (0)$$

(b) [8] Show that the generators $2T_2$, $2T_5$, $2T_7$ form an SU(2) subgroup; that is, show that $\left[2T_2, 2T_5\right] = i2T_7$, etc. To save time, only do two of the three commutators. How does the 3 representation of SU(3) break into representations under this subgroup?

We start exactly the same way, doing the commutation relations.

$$\left[2T_2, 2T_5\right] = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = i2T_7,$$

$$\left[2T_5, 2T_7\right] = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = i2T_2,$$

$$\left[2T_7, 2T_2\right] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = i2T_5.$$

It worked! This time, though, we need to work a bit harder to get the eigenvalues of "T_3", which in this context is $2T_7$. For lack of imagination, we simply take the corresponding determinant, which is

$$0 = \left|2T_7 - \lambda\mathbf{1}\right| = \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & -i \\ 0 & i & -\lambda \end{vmatrix} = -\lambda^3 + \lambda = -\lambda\left(\lambda^2 - 1\right)$$

The eigenvalues are therefore $\{1, -1, 0\}$ which is exactly the weights of the (1) representation.

$$3 \to (1)$$

2. [10] Of the eight generators, two of them can be diagonalized simultaneously (normally chosen as T_3 and T_8). In this problem, you will organize the others into pairs, comparable to the "raising" and "lowering" operators for SU(2)

(a) [5] Combine the remaining six generators, such that the commutation relations of the resulting combinations with T_3 and T_8 always come out proportional to the resulting generators. Here is one of them done for you:

$$T_A = T_1 + iT_2, \quad \text{then} \quad \left[T_3, T_A\right] = +1T_A \quad \text{and} \quad \left[T_8, T_A\right] = 0T_A$$

We can pretty much guess how we want to define all of these things:

$$T_B = T_1 - iT_2, \quad T_C = T_4 + iT_5, \quad T_D = T_4 - iT_5 \quad T_E = T_6 + iT_7, \quad T_F = T_6 - iT_7$$

It is then straightforward to work out the ten commutators:

$$\left[T_3, T_B\right] = -T_B, \quad \left[T_3, T_C\right] = \tfrac{1}{2}T_C, \quad \left[T_3, T_D\right] = -\tfrac{1}{2}T_D, \quad \left[T_3, T_E\right] = -\tfrac{1}{2}T_E, \quad \left[T_3, T_F\right] = \tfrac{1}{2}T_F,$$
$$\left[T_8, T_B\right] = 0T_B, \quad \left[T_8, T_C\right] = \tfrac{\sqrt{3}}{2}T_C, \quad \left[T_8, T_D\right] = -\tfrac{\sqrt{3}}{2}T_C, \quad \left[T_3, T_E\right] = \tfrac{\sqrt{3}}{2}T_E, \quad \left[T_8, T_F\right] = -\tfrac{\sqrt{3}}{2}T_F.$$

(b) [5] For each of the six generators you just worked out, plot on a 2D graph the resulting coefficients when you commute with T_3 and T_8. The first one is done for you. This diagram is called a *root diagram*.

(comment: technically, a root diagram would also include two zero roots, corresponding to the two generators T_3 and T_8 themselves, which commute with each other)

The sketch appears at right. The six points form a perfect regular hexagon. If the double root at zero is added, it would be the same as the weights of the 8 representation, which is not a coincidence.