

## I. Continuous Groups

Up to now we have focused almost exclusively, on finite groups. However, there are many examples of infinite groups that will hold our attention for the second half of this course. We will focus on continuous, connected, compact, finite-dimensional groups, because many of the ideas we have already discussed will work as well on such groups. These terms will be defined as we go along.

### A. Examples of Infinite Groups and Basic Concepts

A group  $\mathcal{G} = \{R\}$  is a set of objects and an operation  $\cdot$  that is closed, associative, has an identity  $E$ , and every element  $R$  has an inverse  $R^{-1}$ . Some examples of infinite groups are (dimensions appear afterwards, in square brackets):

- $\mathbb{R} - \{0\}$ , the real numbers under multiplication, excluding zero. [1]
- $\mathbb{C} - \{0\}$ , the complex numbers under multiplication, excluding zero. The

complex numbers are numbers of the form  $z = x + iy$ , where  $i$  is a square root of minus one. A partial multiplication table is given at right. The complex conjugate is given by  $z^* = x - iy$ . [2]

$\cdot$	<b>1</b>	$i$
<b>1</b>	1	$i$
$i$	$i$	-1

- $\mathbb{H} - \{0\}$ , the quaternions under multiplication, excluding zero. The quaternions are numbers of the form  $z = x + iy + ju + kv$ , where  $i, j$ , and  $k$  are square roots of minus one. A partial multiplication table is given at right. Note that it is not commutative, since

$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j$$

$\cdot$	1	$i$	$j$	$k$
<b>1</b>	1	$i$	$j$	$k$
$i$	$i$	-1	$k$	$-j$
$j$	$j$	$-k$	-1	$i$
$k$	$k$	$j$	$-i$	-1

The quaternionic conjugate of  $z$  is given by  $\bar{z} = x - iy - ju - kv$ . [4]

- $O(n)$ , the orthogonal matrices, the set of  $n \times n$  real matrices  $R$  satisfying  $R^T R = \mathbf{1}$ . These matrices represent all rotations and reflections in  $n$ -dimensional space. [ $n(n-1)/2$ ]
- $SO(n)$ , the special orthogonal matrices, the same as  $O(n)$ , but restricted to rotations with determinant one,  $\det(R) = 1$ . [ $n(n-1)/2$ ]
- $U(n)$ , the unitary matrices, the set of  $n \times n$  complex matrices  $U$  satisfying  $U^\dagger U = \mathbf{1}$ , where  $U^\dagger = (U^*)^T$ . [ $n^2$ ]
- $SU(n)$ , the special unitary matrices, the same as  $U(n)$ , with the additional restriction  $\det(U) = 1$ . [ $n^2 - 1$ ]
- $Sp(n)$ , the symplectic matrices, the set of  $n \times n$  quaternionic matrices  $V$  satisfying  $V^\dagger V = \mathbf{1}$ , where  $V^\dagger = (\bar{V})^T$ . [ $n(2n-1)$ ]

Given any two groups  $\mathcal{G} = \{R\}$  and  $\mathcal{H} = \{S\}$ , we can define the *direct product* of the two groups  $\mathcal{G} \times \mathcal{H}$  as the set of elements  $\{(R, S)\}$  with multiplication rule

$$(R, S) \cdot (R', S') = (R \cdot R', S \cdot S'). \quad (1.1)$$

The group  $\mathcal{G} \times \mathcal{H}$  contains both  $\mathcal{G}$  and  $\mathcal{H}$  as subgroups, because the elements of the form  $(R, E)$  act like elements of  $\mathcal{G}$  and the elements  $(E, S)$  act like elements of  $\mathcal{H}$ . We therefore often abbreviate  $(R, E) = R$  and  $(E, S) = S$ , at least when there is no confusion, and therefore all elements can be written as  $(R, S) = RS$ . It is easy to see that elements in  $\mathcal{G}$  commute with those in  $\mathcal{H}$ , because

$$R \cdot S = (R, E) \cdot (E, S) = (R, S) = (E, S) \cdot (R, E) = S \cdot R \quad (1.2)$$

The direct products  $\mathcal{G} \times \mathcal{H}$  and  $\mathcal{H} \times \mathcal{G}$  are essentially the same, and we will treat them as identical. It is also possible to take direct products of more than two groups,  $\mathcal{G} \times \mathcal{H} \times \mathcal{K}$ .

In fact, these examples are a nearly complete list of all the groups we will be considering in the second half of this course. All of our examples will be no more complicated than the direct products of these groups, plus five other groups we will not encounter until much later (and will do virtually no work with).

The first problem when encountering such infinite groups lies in naming the infinite number of elements. Imagine designating each element of the group by *coordinates*, some list of real numbers,  $\mathbf{x} = (x_1, x_2, \dots, x_N)$ , and we associate some element of the group  $R$  with this list, so we have

$$R = R(\mathbf{x}) = R(x_1, x_2, \dots, x_N) \quad (1.3)$$

It will be assumed that this mapping is such that every element of  $\mathcal{G}$  can be uniquely denoted by its coordinates. For example, for the complex numbers, we might choose  $R$  to be  $R(x_1, x_2) = x_1 + ix_2$ , or we might equally well denote complex numbers by

$R(x_1, x_2) = x_1 e^{ix_2}$ . For the set of rotations in 3D space, we could denote all rotations by the three Euler angles,  $R(\alpha, \beta, \gamma)$ , for example. Note that in some cases, we can get all the elements of the group by using only a restricted range of real numbers; for example, if we denote complex numbers by  $x_1 e^{ix_2}$ , we will find that we need only consider  $x_1$  positive and  $x_2$  restricted to the range  $-\pi \leq x_2 \leq \pi$ . There will be some special coordinate  $\mathbf{e}$  (often, but not always, chosen as the origin) that corresponds to the identity element  $E$ , so

$$\mathbf{e} = R^{-1}(E) \Leftrightarrow E = R(\mathbf{e}). \quad (1.4)$$

For the complex numbers with  $R(x_1, x_2) = x_1 + ix_2$ , for example,  $\mathbf{e} = (1, 0)$ .

We also want our coordinates to, in some sense, be “continuous,” by which we mean, that group elements that are near each other are designated by coordinates that are near each other. Formally, this requires defining a “topology” (definition of nearby) on

the group elements, but we won't worry about this. It actually turns out that the Euler angles are *not* a good choice of coordinates for this reason. It can be shown that to perform a rotation around the  $x$ -axis by a small angle  $\theta$ , you must use Euler angles  $R(\frac{1}{2}\pi, \theta, -\frac{1}{2}\pi)$ , even though this rotation is very close to  $R(0, 0, 0) = E$ . Other coordinates, such as the ones we will encounter in section D, avoid this problem.

A *continuous* group is then one in which nearby elements can be mapped to nearby coordinates. The number of coordinates required to denote elements of the group is called the *dimension*  $N$  of the group. For example, complex numbers and quaternions have dimension 2 and 4 respectively, since it takes 2 or 4 real numbers to specify them. A group is *finite dimensional* if only a finite number of coordinates is needed. All of the examples listed above are finite dimensional; the dimensions are listed in square brackets after the group description.

It will often be awkward to perform multiplication using the abstract group elements. We define the coordinate product function  $\mu(\mathbf{x}, \mathbf{y})$  by

$$\mu(\mathbf{x}, \mathbf{y}) \equiv R^{-1}(R(\mathbf{x}) \cdot R(\mathbf{y})), \quad (1.5)$$

where  $R^{-1}$  is the inverse of  $R$ , which turns group elements into coordinates. In other words, multiply the group element corresponding to  $\mathbf{x}$  and the group element corresponding to  $\mathbf{y}$  and figure out the coordinates of the resulting group element. As an example, suppose we are working with the complex numbers, and we denote them by a pair of coordinates  $R(x_1, x_2) = x_1 + ix_2$ . Then we would have

$$\begin{aligned} \mu(\mathbf{x}, \mathbf{y}) &= \mu(x_1, x_2; y_1, y_2) = R^{-1}[R(x_1, x_2) \cdot R(y_1, y_2)] = R^{-1}[(x_1 + ix_2)(y_1 + iy_2)] \\ &= R^{-1}[(x_1y_1 - x_2y_2) + i(x_1y_2 + x_2y_1)] = (x_1y_1 - x_2y_2, x_1y_2 + x_2y_1) \end{aligned} \quad (1.6)$$

From the associative and identity properties of the group, it is easily proven that

$$\mu(\mathbf{x}, \mathbf{e}) = \mu(\mathbf{e}, \mathbf{x}) = \mathbf{x}, \quad (1.7a)$$

$$\mu(\mathbf{x}, \mu(\mathbf{y}, \mathbf{z})) = \mu(\mu(\mathbf{x}, \mathbf{y}), \mathbf{z}) \quad (1.7b)$$

Actually finding the *form* of these coordinate multiplications will generally not concern us; we will need them only for proofs.

The last definition we need in this section is *connected*. A group is connected if every element in it can be built out of small elements close to the identity element  $E$ . For example, the proper rotations  $SO(3)$  is connected, since you can perform any rotation with a finite number of small rotations, but  $O(3)$  is not, because you can't, for example, perform inversion by making many small rotations. Of the groups listed, only  $\mathbb{R} - \{0\}$  and  $O(n)$  are disconnected, the rest are connected.

## B. Measure and Compactness

We would like to use many of our theorems from finite groups; for example, the rearrangement theorem tells us that if  $x$  is any element of the group, and if we multiply on the right (or left) by every other element of the group, we get every element back exactly once. We used this most commonly in the form

$$\sum_{R \in \mathcal{G}} f(R) = \sum_{R \in \mathcal{G}} f(RS) = \sum_{R \in \mathcal{G}} f(SR) \quad (1.8)$$

for any element  $S$  of the group, and  $f$  is any function of the group elements. The problem is that for infinite groups, the sum (1.8) may well be infinite (it usually will be), and we don't know what to make of it. The solution is to replace the sums by integrals, so we would like something like

$$\int f(R) dR = \int f(RS) dR = \int f(SR) dR, \quad (1.9)$$

However, we don't know what (1.9) means; we need to define it. We don't know how to integrate over abstract group elements, but we *do* know how to integrate over coordinates. A naive interpretation of integration might then be to define

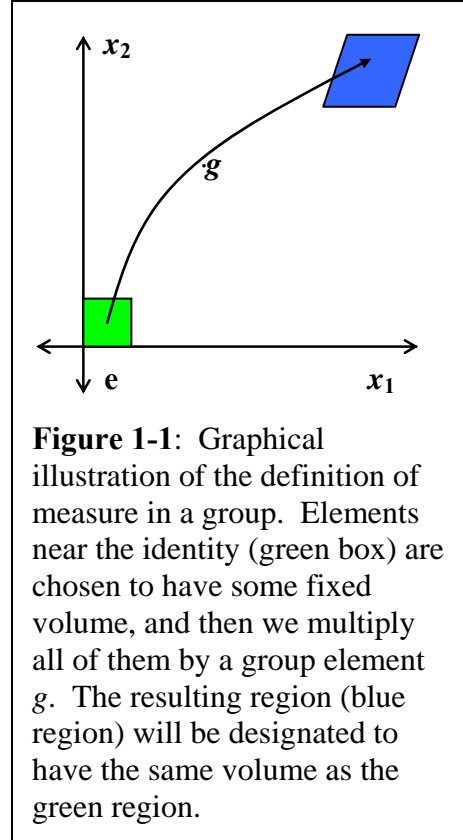
$$\int f(R) dR_{\text{Naive}} = \int d^N \mathbf{x} f(R(\mathbf{x})), \quad (1.10)$$

however, this lacks the desired properties (1.9). The basic problem is that coordinates can be defined in a completely arbitrary way, and consequently it is quite possible to have group elements that are relatively nearby designated by coordinates that are quite spread out, or vice versa, so that the naive integral (1.10) exaggerates the influence of one region, while suppressing that of another. The correct form can be worked out by taking a small region near  $E$ , and arbitrarily defining its volume, and then multiplying this small volume by an arbitrary element  $R$  and demanding that the new region have the same volume, as sketched in Fig. 1-1. This leaves ambiguous whether we should multiply the volume by  $R$  on the right or on the left, and it turns out we will end up needing *two* different measures, called the left and right measure, defined by

$$\int d_R R f(R) = C \int d^N \mathbf{x} \left| \partial \mu_i(\mathbf{y}, \mathbf{x}) / \partial y_j \right|_{\mathbf{y}=\mathbf{e}}^{-1} f(R(\mathbf{x})), \quad (1.11a)$$

$$\int d_L R f(R) = C \int d^N \mathbf{x} \left| \partial \mu_i(\mathbf{x}, \mathbf{y}) / \partial y_j \right|_{\mathbf{y}=\mathbf{e}}^{-1} f(R(\mathbf{x})). \quad (1.11b)$$

where  $C$  is an arbitrary constant. I now assert that each of these has one of the desired properties (1.9). I will demonstrate it explicitly for the left measure.



Let  $S$  be any element of the group, then  $S = R(\mathbf{s})$  for some coordinate  $\mathbf{s}$ . We wish to simplify the expression

$$\begin{aligned}\int d_L R f(S \cdot R) &= C \int d^N \mathbf{x} \left| \frac{\partial \mu_i(\mathbf{x}, \mathbf{y})}{\partial y_j} \right|_{\mathbf{y}=\mathbf{e}}^{-1} f(R(\mathbf{s}) \cdot R(\mathbf{x})) \\ &= C \int d^N \mathbf{x} \left| \frac{\partial \mu_i(\mathbf{x}, \mathbf{y})}{\partial y_j} \right|_{\mathbf{y}=\mathbf{e}}^{-1} f(R(\boldsymbol{\mu}(\mathbf{s}, \mathbf{x})))\end{aligned}\tag{1.12}$$

Now, change variables on the right side. Let  $\mathbf{z} = \boldsymbol{\mu}(\mathbf{s}, \mathbf{x})$ , then as  $\mathbf{x}$  varies over all possible coordinates of the group, so will  $\mathbf{z}$ , so we can replace the integral over all  $\mathbf{x}$  with the integral over all  $\mathbf{z}$ . However, when performing such a change of variables, we know from multi-variable calculus that a Jacobian must be included in the integral, so that

$$\int d^N \mathbf{z} = \int d^N \mathbf{x} \left| \frac{\partial z_i}{\partial x_j} \right| = \int d^N \mathbf{x} \left| \frac{\partial \mu_i(\mathbf{s}, \mathbf{x})}{\partial x_j} \right|\tag{1.13}$$

This allows us to rewrite (1.12) as

$$\begin{aligned}\int d_L R f(S \cdot R) &= C \int d^N \mathbf{z} \left| \frac{\partial \mu_i(\mathbf{s}, \mathbf{x})}{\partial x_j} \right|^{-1} \left| \frac{\partial \mu_i(\mathbf{x}, \mathbf{y})}{\partial y_j} \right|_{\mathbf{y}=\mathbf{e}}^{-1} f(R(\mathbf{z})) \\ &= C \int d^N \mathbf{z} \left| \sum_k \frac{\partial \mu_i(\mathbf{s}, \mathbf{x})}{\partial x_k} \frac{\partial \mu_k(\mathbf{x}, \mathbf{y})}{\partial y_j} \right|_{\mathbf{y}=\mathbf{e}}^{-1} f(R(\mathbf{z}))\end{aligned}\tag{1.14}$$

where we have used the fact that the product of a determinant of two matrices equals the determinant of the product. Now, consider the expression

$$\begin{aligned}\left. \frac{\partial \mu_i(\mathbf{z}, \mathbf{y})}{\partial y_j} \right|_{\mathbf{y}=\mathbf{e}} &= \left. \frac{\partial \mu_i(\boldsymbol{\mu}(\mathbf{s}, \mathbf{x}), \mathbf{y})}{\partial y_j} \right|_{\mathbf{y}=\mathbf{e}} = \left. \frac{\partial \mu_i(\mathbf{s}, \boldsymbol{\mu}(\mathbf{x}, \mathbf{y}))}{\partial y_j} \right|_{\mathbf{y}=\mathbf{e}} \\ &= \sum_k \left. \frac{\partial \mu_i(\mathbf{s}, \mathbf{v})}{\partial v_k} \right|_{\mathbf{v}=\boldsymbol{\mu}(\mathbf{x}, \mathbf{e})} \left. \frac{\partial \mu_k(\mathbf{x}, \mathbf{y})}{\partial y_j} \right|_{\mathbf{y}=\mathbf{e}} = \sum_k \left. \frac{\partial \mu_i(\mathbf{s}, \mathbf{x})}{\partial x_k} \frac{\partial \mu_k(\mathbf{x}, \mathbf{y})}{\partial y_j} \right|_{\mathbf{y}=\mathbf{e}}\end{aligned}\tag{1.16}$$

Substituting this in (1.15), and then comparing the result with (1.11b), we see that

$$\int d_L R f(S \cdot R) = C \int d^N \mathbf{z} \left| \frac{\partial \mu_i(\mathbf{z}, \mathbf{y})}{\partial y_j} \right|_{\mathbf{y}=\mathbf{e}}^{-1} f(R(\mathbf{z})) = \int d_L R f(R)\tag{1.17}$$

A nearly identical proof then shows that

$$\int d_R R f(R \cdot S) = \int d_R R f(R)\tag{1.18}$$

It can further be proven that the definitions (1.11) are independent of the choice of coordinates, although a different constant  $C$  must generally be chosen.

To illustrate how measure is defined, consider the complex numbers, with the standard coordinates so that  $R(x_1, x_2) = x_1 + ix_2$ . Then with the help of (1.6), we find

$$\left| \frac{\partial \mu_i(\mathbf{x}, \mathbf{y})}{\partial y_j} \right|_{\mathbf{y}=\mathbf{e}} = \begin{vmatrix} x_1 & -x_2 \\ x_2 & x_1 \end{vmatrix}_{\mathbf{y}=(1,0)} = x_1^2 + x_2^2 \quad (1.19)$$

and we can use this to demonstrate that the left measure is given by (choosing  $C = 1$ ):

$$\int d_L z f(z) = \int \frac{dx_1 dx_2}{x_1^2 + x_2^2} f(x_1 + ix_2) \quad (1.20)$$

The right measure turns out to be identical (no surprise, since the group is commutative), but this will not generally be the case.

In some cases, the left and right measure *are* identical. Consider the integral

$$\int d_R S \int d_L R f(SR) = \int d_R S \int d_L R f(R) = \int d_R S \int d_L R f(S) \quad (1.21)$$

where we have used (1.17) and (1.18) to remove *either*  $S$  or  $R$  from the integration. For each of the expressions, there is one integral that is trivial, since it no longer involves the function  $f$ . Define the (left and right) volume of the group as

$$V_R = \int d_R R, \quad V_L = \int d_L R \quad (1.22)$$

then it is easy to see that (1.21) becomes

$$V_R \int d_L R f(R) = V_L \int d_R R f(R) \quad (1.23)$$

where we have changed our elements of the group to  $R$ . Now, *provided* the volumes on each side are finite, we realize that left and right integration are always proportional. Indeed, if you use a function that is non-zero only very near the identity element, it is not hard to show from the definitions (1.11) that the proportionality constant is one. This implies that the two volumes must be the same as well, so, provided  $V$  is finite, we have

$$\int d_R R = \int d_L R \equiv V \quad (1.24a)$$

$$\int d_L R f(R) = \int d_R R f(R) \equiv \int dR f(R) \quad (1.24b)$$

Equation (1.24a) has been proven only in the case of finite volume, but simple arguments show that if either integral is infinite, both integrals will be infinite. However, (1.24b) works only for finite volume groups, and in this case we may use (1.9).

We are now ready to define *compact* groups: A continuous group is compact if the volume of the group is finite. As we will demonstrate below, many of our theorems we previously developed can be applied to compact groups as well.

Intuitively, it is not hard to understand the concept of compactness. The group  $SO(3)$  is compact, because we can find a finite list of rotations such that every possible rotation is within, say, one degree of our list. In contrast, no finite list of real numbers can be made such that every real number is within a factor of 1.1 of one of the numbers. That the real numbers (or complex, or quaternions) have infinite volume can be verified directly using (1.25a). Fortunately, we won't be using this definition, since soon we'll have a better way. In fact, of our examples we gave in the first section, only the first three have infinite volume; all the others are compact.

### C. Representations

In a manner similar to before, we define a representation  $\Gamma(R)$  as a set of  $l \times l$  invertible matrices satisfying

$$\Gamma(R)\Gamma(R') = \Gamma(R \cdot R') \quad (1.25)$$

The number  $l$  is the *dimension* of the representation, and in general has no connection with the dimension of the group. These matrices could, in principle, be real, complex, or quaternionic, but we will focus on complex matrices (real matrices being a special case of complex). A representation may have more than one element of the group represented by the same matrix; when this does *not* occur, we say the representation is *faithful*, but we won't attach any significance to faithful representations.

Given some representations, there are several ways to make some new representations:

- **Similarity:** Let  $\Gamma$  be a representation, and  $S$  a constant invertible matrix of the same size, then we can make a new representation of the form

$$\Gamma'(R) = S^{-1}\Gamma(R)S \quad (1.26)$$

This representation has the same dimension as  $\Gamma(R)$ . When two such representations are related by (1.26), we say they are similar, and write  $\Gamma' \sim \Gamma$ .

- **Complex Conjugate:** Let  $\Gamma$  be a representation, then the complex conjugate representation  $\Gamma^*$  produced simply by taking the complex conjugate of every component is also a representation. This representation has the same dimension as  $\Gamma$ .
- **Direct Sum:** Let  $\Gamma^A$  and  $\Gamma^B$  be two representations, then define  $\Gamma^{A \oplus B}$  as

$$\Gamma^{A \oplus B}(R) = \begin{pmatrix} \Gamma^A(R) & 0 \\ 0 & \Gamma^B(R) \end{pmatrix} \quad (1.27)$$

If the dimensions of  $\Gamma^A$  and  $\Gamma^B$  are  $l_A$  and  $l_B$  respectively, then the dimension of  $\Gamma^{A \oplus B}$  is  $l_A + l_B$ .

- **Tensor Product:** Let  $\Gamma^A$  and  $\Gamma^B$  be two representations, then define  $\Gamma^{A \otimes B}$  by  $\Gamma^{A \otimes B}(R) = \Gamma^A(R) \otimes \Gamma^B(R)$ , defined by its components:

$$\left[ \Gamma^A(R) \otimes \Gamma^B(R) \right]_{ij, mn} = \Gamma_{im}^A(R) \Gamma_{jn}^B(R) \quad (1.28)$$

where the indices  $i$  and  $m$  run from 1 to  $l_A$ , and  $j$  and  $n$  run from 1 to  $l_B$ . The indices on the tensor product representation are understood to not be multiplied, but rather, represent pairs of indices. The dimension of  $\Gamma^{A \otimes B}$  is  $l_A l_B$ .

It is easy to show that the similarity relationship has all the properties of an equivalence relation; that is  $\Gamma \sim \Gamma$ , if  $\Gamma \sim \Gamma'$  then  $\Gamma' \sim \Gamma$ , and if  $\Gamma \sim \Gamma'$  and  $\Gamma' \sim \Gamma''$  then  $\Gamma \sim \Gamma''$ . This means that representations can be grouped together based on similarity, and

we will in fact treat similar representations as if they are identical. It is easily demonstrated that  $\Gamma^{A\oplus B} \sim \Gamma^{B\oplus A}$  and  $\Gamma^{A\otimes B} \sim \Gamma^{B\otimes A}$

If a representation is explicitly real ( $\Gamma(R)$  real for all  $R$ ), or if it is equivalent to such a representation, we will say that the representation is *real*. Sometimes, it turns out that it is impossible to make  $\Gamma(R)$  real, even though  $\Gamma^* \sim \Gamma$ ; in this case we call the representation *pseudoreal*. Sometimes, the complex conjugate is simply inequivalent to the representation, in which case we call the representation *complex*.

If a representation is in block-diagonal form, it can obviously be written as a direct sum. If it is block-diagonal, or equivalent to block-diagonal, we call the representation *reducible*. If it is not reducible, we call it *irreducible*. As an abbreviation, an irreducible representation will be called an *irrep*. Much of our work will be in trying to find and characterize the irreps of our various groups.

It's now time to start proving a few theorems that we derived for finite groups. The first and most important is the fact that representations of *compact* groups are always equivalent to unitary representations of the groups. The proofs are virtually identical to those given by Dr. Holzwarth in the first half of the course, except that sums over group elements are always replaced by integrals.

Let  $\Gamma(R)$  be a representation of a compact group. Define

$$A = \int dR \Gamma^\dagger(R) \Gamma(R) \quad (1.29)$$

This matrix is manifestly Hermitian, and therefore can be written in the form  $A = VDV^\dagger$ , where  $V$  is unitary, and  $D$  is real and diagonal. Furthermore,  $A$  is positive definite, because

$$\langle \psi | A | \psi \rangle = \int dR \langle \psi | \Gamma^\dagger(R) \Gamma(R) | \psi \rangle = \int dR |\Gamma(R) | \psi \rangle|^2 > 0 \quad (1.30)$$

It follows that  $A$  has positive eigenvalues, so  $D$  has positive numbers along its diagonal, and therefore  $D^{1/2}$  and  $D^{-1/2}$  both exist. We define the matrix  $S$  by

$$S = VD^{-1/2}, \quad S^{-1} = D^{1/2}V^\dagger \quad (1.31)$$

and define a new representation similar to the old

$$\Gamma'(R) = S^{-1} \Gamma(R) S = D^{1/2} V^\dagger \Gamma(R) V D^{-1/2}. \quad (1.32)$$

I now assert that the new representation is unitary. We demonstrate this explicitly:

$$\begin{aligned} \Gamma'(S)^\dagger \Gamma'(S) &= D^{-1/2} V^\dagger \Gamma^\dagger(S) V D^{1/2} D^{1/2} V^\dagger \Gamma(S) V D^{-1/2} = D^{-1/2} V^\dagger \Gamma^\dagger(S) A \Gamma(S) V D^{-1/2} \\ &= D^{-1/2} V^\dagger \Gamma^\dagger(S) \left[ \int dR \Gamma^\dagger(R) \Gamma(R) \right] \Gamma(S) V D^{-1/2} \\ &= D^{-1/2} V^\dagger \left[ \int dR \Gamma^\dagger(RS) \Gamma(RS) \right] V D^{-1/2} = D^{-1/2} V^\dagger \left[ \int dR \Gamma^\dagger(R) \Gamma(R) \right] V D^{-1/2} \\ &= D^{-1/2} V^\dagger A V D^{-1/2} = D^{-1/2} V^\dagger V D V^\dagger V D^{-1/2} = D^{-1/2} D D^{-1/2} = 1 \end{aligned} \quad (1.33)$$

We therefore can focus exclusively on unitary representations. Fortunately, for nearly all the groups we are working with, the definition of the group itself provides a unitary representation. The first three examples from section A were not compact, so we won't



worry about them, and all of the other examples except  $Sp(n)$  are defined in terms of unitary matrices. Hence we can skip the step of proving they are compact.

Other theorems about representations follow in a similar manner. For example, if we have a complete list of the inequivalent unitary irreps of a group,  $\{\Gamma^{(a)}(R)\}$ , then we can prove the great orthogonality theorem:

$$\int dR \Gamma_{ij}^A(R) \Gamma_{mn}^{B\dagger}(R) = \delta_{im} \delta_{jn} \delta_{AB} V / l_a \quad (1.34)$$

where  $V$  is the volume of the group, and  $l_a$  is the dimension of one of the representations.

As with finite groups, it is often useful to work with the character, defined by

$$\chi(R) = \text{tr}(\Gamma(R)) = \sum_i \Gamma_{ii}(R) \quad (1.35)$$

Similar representations have equal characters. The characters for the other ways of creating new representations are similarly simply related.

$$\chi^{A*}(R) = \chi^A(R)^* \quad (1.36a)$$

$$\chi^{A \oplus B}(R) = \chi^A(R) + \chi^B(R) \quad (1.36b)$$

$$\chi^{A \otimes B}(R) = \chi^A(R) \chi^B(R). \quad (1.36c)$$

From (1.34) we can then demonstrate

$$\int dR \chi^A(R) \chi^B(R)^* = \delta_{ab} V \quad (1.37)$$

Some other expressions that work well for finite groups are unfortunately not very helpful with infinite groups. For example, it is still true that the sum of the squares of the dimensions of the irreps of a group adds up to the order of the group, but since the group has an infinite number of elements, this simply tells you that there are an infinite number of irreps.

When we deal with direct products of groups, it turns out that finding irreps is pretty straightforward. Suppose the irreps of  $\mathcal{G} = \{R\}$  are  $\{\Gamma^A(R)\}$ , and of  $\mathcal{H} = \{R'\}$  are  $\{\Gamma'^B(R')\}$ . Then the irreps of  $\mathcal{G} \times \mathcal{H}$  will be of the tensor product-like form

$$\Gamma^{A,B}(R, R') = \Gamma^A(R) \otimes \Gamma'^B(R') \quad (1.38)$$

Their characters are simply related as well.

$$\chi^{A,B}(R, R') = \chi^A(R) \chi'^B(R') \quad (1.39)$$

Hence when working out irreps, we need not work them out for direct products of groups; if we know them for  $\mathcal{G}$  and  $\mathcal{H}$  we know them for  $\mathcal{G} \times \mathcal{H}$ .

Although formulas like (1.35) and (1.38) are elegant and sometimes useful, they are unwieldy because of the complications of actually performing the integrals. Fortunately, for compact connected groups, we can focus on elements near the identity  $E$ , and this will significantly reduce our work. We turn now to the subject of generators.

## D. Generators of a Group

Because the groups we are considering are infinite, it is difficult to discuss all the elements of the group. In both finite and infinite groups, it is commonplace to simply describe some of the elements. For example, for the finite group  $C_N$  we discussed in the previous half of the course, all elements can be built out of a single element  $C_N$  raised to various powers. It is sufficient, therefore, when describing a representation of the group  $C_N$ , to give only one matrix  $\Gamma(C_N)$ . For continuous connected compact groups, we need only consider elements very near the identity  $E$ , since all elements are products of such elements.

Suppose we choose coordinates such that  $R(0) = E$ ; that is, the coordinate associated with the identity is  $\mathbf{e} = 0$ . Then let  $\Gamma(R)$  be a unitary representation. We always map the identity element to the identity matrix  $\Gamma(E) = \mathbf{1}$ . We define the *generators*  $T_a$  of this representation as the matrices

$$T_a = i \frac{\partial}{\partial x_a} \Gamma(R(\mathbf{x}))_{\mathbf{x}=0} \quad (1.39)$$

Obviously, the number of generators is equal to the dimension of the group. For elements very close to the identity element, we can always Taylor expand our elements about  $E$ . To linear order in  $\mathbf{x}$ ,

$$\Gamma(R(\mathbf{x})) = \Gamma(R(0)) - i\mathbf{T} \cdot \mathbf{x} = \mathbf{1} - i\mathbf{T} \cdot \mathbf{x} = \exp(-i\mathbf{T} \cdot \mathbf{x}) \quad (1.40)$$

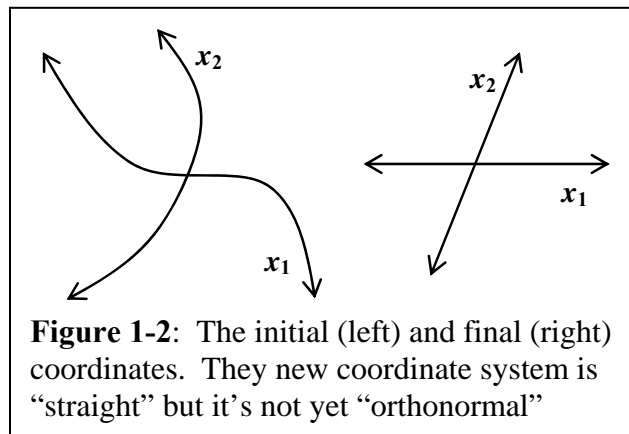
The demand that  $\Gamma(R)$  be unitary then requires, to linear order

$$\mathbf{1} = \Gamma^\dagger(R(\mathbf{x}))\Gamma(R(\mathbf{x})) = (\mathbf{1} - i\mathbf{T} \cdot \mathbf{x})^\dagger (\mathbf{1} - i\mathbf{T} \cdot \mathbf{x}) = \mathbf{1} + i\mathbf{x} \cdot (\mathbf{T}^\dagger - \mathbf{T}) \quad (1.41)$$

Hence the generators must be Hermitian,

$$\mathbf{T}^\dagger = \mathbf{T} \quad (1.42)$$

Up to now we have allowed our coordinates to be completely arbitrary, other than the insistence that they be smooth. We now wish to change our coordinates in a more sensible manner. Let  $\mathbf{x}$  be an arbitrary small coordinate, but let  $M$  be an integer, such that  $M\mathbf{x}$  can be made as large as we wish. Then let's define the group element corresponding to  $M\mathbf{x}$  to be that which occurs if we produce  $M$  small steps of size  $\mathbf{x}$ . In other words,  $R(M\mathbf{x}) = R(\mathbf{x})^M$ . We are, in a sense, shifting from coordinates that initially might have been very "curved" to ones that now go "straight" away from the origin, as sketched in Fig. 1-2. Then we



see that for such large coordinates, we have

$$\Gamma(R(M\mathbf{x})) = \Gamma(R(\mathbf{x})^M) = \Gamma(R(\mathbf{x}))^M = [\exp(-i\mathbf{T} \cdot \mathbf{x})]^M = \exp(-i\mathbf{T} \cdot M\mathbf{x}) \quad (1.43)$$

Since  $M\mathbf{x}$  can be a large coordinate, we may rewrite (1.43) setting  $M\mathbf{x} \rightarrow \mathbf{x}$ , but  $\mathbf{x}$  is no longer restricted to be small, so we have

$$\Gamma(R(\mathbf{x})) = \exp(-i\mathbf{T} \cdot \mathbf{x}) \quad (1.44)$$

Our coordinates are now “straight” but not “orthogonal.” Consider the matrix

$$M_{ab} = \text{tr}(T_a T_b) \quad (1.45)$$

Keeping in mind that the generators are Hermitian and the cyclic property of the trace, it is easy to show that  $M$  is a real, positive symmetric matrix. It can therefore be diagonalized by creating new generators  $T'_a$  that are linear combinations of the old. One can similarly define new coordinates  $\mathbf{x}'$  that are linear combinations of the old, such that expression (1.44) takes the form  $\Gamma(R(\mathbf{x})) = \exp(-i\mathbf{T}' \cdot \mathbf{x}')$ . The matrix  $M$  in this new basis will now be diagonal, so

$$\text{tr}(T'_a T'_b) = k_a \delta_{ab} \quad (1.46)$$

We can then multiply each  $T'_a$  by an arbitrary constant to make new generators  $T''_a$  with whatever trace we want, so now  $\text{tr}(T''_a T''_b) = \lambda \delta_{ab}$ . where we get to pick  $\lambda$ . We will then discard the original generators  $T_a$  and the first modified generators  $T'_a$  in favor of  $T''_a$ . We will, however, never use the unmodified generators again, so to avoid unnecessary adornments, we will simply call these newest generators  $T_a$ , and drop the double primes. So we have

$$\text{tr}(T_a T_b) = \lambda \delta_{ab} \quad (1.47)$$

Consider now the following product, which we expand out to second order in the coordinates:

$$\begin{aligned} \Gamma(R(\mathbf{x})R(\mathbf{y})) &= \exp(-i\mathbf{T} \cdot \mathbf{x}) \exp(-i\mathbf{T} \cdot \mathbf{y}) \\ &= 1 - i(\mathbf{T} \cdot \mathbf{x} + \mathbf{T} \cdot \mathbf{y}) + \frac{1}{2}i^2(\mathbf{T} \cdot \mathbf{x} + \mathbf{T} \cdot \mathbf{y})^2 + \frac{1}{2}i^2[\mathbf{T} \cdot \mathbf{x}, \mathbf{T} \cdot \mathbf{y}], \\ \Gamma(R(\mathbf{x})R(\mathbf{y})) &= \exp\left\{-i(\mathbf{T} \cdot \mathbf{x} + \mathbf{T} \cdot \mathbf{y} - \frac{1}{2}i[\mathbf{T} \cdot \mathbf{x}, \mathbf{T} \cdot \mathbf{y}])\right\} \end{aligned} \quad (1.48)$$

Comparison with (1.40) tells us that the commutator of two  $T$ 's must be another  $T$  (or combinations thereof), so we have

$$[T_a, T_b] = i \sum_c f_{abc} T_c \quad (1.49)$$

The constants  $f_{abc}$  are called *structure constants* and are an inherent property of the group. They will be the same in all representations of the group, because the product appearing in (1.48) must yield the same element for all representations. Indeed, it can be shown that at least in some region around the identity element  $E$ , the group multiplication table

can be completely determined by the structure constants, and hence they represent a very minimal description of the entire group. If you are given the matrices  $T_a$  and need the structure constants, they can be found in a straightforward manner with the help of (1.47):

$$i\lambda f_{abc} = \text{tr}([T_a, T_b]T_c) \quad (1.50)$$

Indeed, with the help of the cyclic property of the trace, it is possible to show that the structure constants are completely anti-symmetric, so that

$$f_{abc} = f_{bca} = f_{cab} = -f_{acb} = -f_{cba} = -f_{bac} \quad (1.51)$$

In addition, they satisfy an additional identity. It is trivial to demonstrate

$$[T_a, [T_b, T_c]] + [T_b, [T_c, T_a]] + [T_c, [T_a, T_b]] = 0 \quad (1.52)$$

by simply writing out all the terms. Using (1.49), this implies

$$\sum_d (f_{bcd}f_{ade} + f_{cad}f_{bde} + f_{abd}f_{cde}) = 0 \quad (1.53)$$

A set of generators satisfying (1.49) are termed a *Lie algebra*, and the groups they generate are called *Lie groups*. Because we demanded that our groups all be compact, we will have Hermitian generators and unitary representations, and the groups we get are called compact Lie groups.

Suppose we find a set of real structure constants  $f$  that are completely anti-symmetric (1.51), and also satisfy (1.53). Is it guaranteed that there are a set of matrices  $T_a$  that will satisfy (1.49)? The answer is yes, by construction. Given the  $N$  structure constants, define a set of  $N \times N$  matrices  $T_a$  defined by its components:

$$(T_a)_{bc} = -if_{abc} \quad (1.54)$$

Then working out the commutators of the  $T_a$ 's with the help of (1.51) and (1.53), it is not hard to show that (1.49) is automatically satisfied. This representation of the group is called the *adjoint representation* and has the same dimension as the dimension of the group,  $l_{\text{adj}} = N$ .

We will sometimes need to find the generators of a representation after we have performed a similarity transformation, a complex conjugation, a direct sum of representations, or a tensor product. The first three of these are pretty easy to work out. If  $S$  is the matrix that relates two representations,  $\Gamma'(R) = S^{-1}\Gamma(R)S$ , then it is not surprising that

$$T'_a = S^{-1}T_a S \quad (1.55)$$

The complex conjugate is only slightly tricky; because of the factor of  $i$  in (1.44), which introduces an extra minus sign.

$$T_a^{A*} = -(T_a^A)^* \quad (1.56)$$

It is easy to demonstrate, for example, that the eigenvalues of the generators of the complex conjugate representations are just the negatives of the eigenvalues of the generators of the representation itself. For the direct sum, you can probably guess:

$$T_a^{A \oplus B} = \begin{pmatrix} T_a^A & 0 \\ 0 & T_a^B \end{pmatrix} \quad (1.57)$$

For the direct product, you probably couldn't:

$$T_a^{A \otimes B} = T_a^A \otimes \mathbf{1}_{l_B} + \mathbf{1}_{l_A} \otimes T_a^B \quad (1.58)$$

where  $\mathbf{1}_l$  is the  $l \times l$  unit matrix. To demonstrate that this is correct, consider the expression

$$\Gamma^{A \otimes B}(R(\mathbf{x})) = \exp(-i\mathbf{x} \cdot \mathbf{T}^{A \otimes B}) = \exp\left[-(i\mathbf{x} \cdot \mathbf{T}^A) \otimes \mathbf{1} - \mathbf{1} \otimes (i\mathbf{x} \cdot \mathbf{T}^B)\right] \quad (1.59)$$

The two terms in the exponential commute, because their product in either order is  $(i\mathbf{x} \cdot \mathbf{T}^A) \otimes (i\mathbf{x} \cdot \mathbf{T}^B)$ . So we can split this into the product of two exponentials, and we get

$$\begin{aligned} \Gamma^{A \otimes B}(R(\mathbf{x})) &= \exp\left[-(i\mathbf{x} \cdot \mathbf{T}^A) \otimes \mathbf{1}\right] \exp\left[\mathbf{1} \otimes (-i\mathbf{x} \cdot \mathbf{T}^B)\right] \\ &= \left[\exp(-i\mathbf{x} \cdot \mathbf{T}^A) \otimes \mathbf{1}\right] \left[\mathbf{1} \otimes \exp(-i\mathbf{x} \cdot \mathbf{T}^B)\right] = \exp(-i\mathbf{x} \cdot \mathbf{T}^A) \otimes \exp(-i\mathbf{x} \cdot \mathbf{T}^B), \\ \Gamma^{A \otimes B}(R(\mathbf{x})) &= \Gamma^A(R(\mathbf{x})) \otimes \Gamma^B(R(\mathbf{x})) \end{aligned} \quad (1.60)$$

From a practical standpoint, the effect of (1.58) is that if you know the  $l_A$  eigenvalues of  $T_a^A$  and the  $l_B$  eigenvalues of  $T_a^B$ , the eigenvalues of  $T_a^{A \otimes B}$  will be the sums of any one eigenvalue of each, for a total of  $l_A l_B$  combinations.

It is also helpful to know how the generators of a direct product of groups works out. If  $N$  and  $M$  are the dimensions of the groups  $\mathcal{G}$  and  $\mathcal{H}$ , then the dimension of  $\mathcal{G} \times \mathcal{H}$  will be  $N + M$ . There will be one generator of  $\mathcal{G} \times \mathcal{H}$  for every generator of  $\mathcal{G}$ , and also for every generator of  $\mathcal{H}$ . If the generators of  $\mathcal{G}$  and  $\mathcal{H}$  are  $T_a^A$  and  $T_b^B$ , then the corresponding generators of the representations of  $\mathcal{G} \times \mathcal{H}$  will be

$$T_a^{A,B} = T_a^A \otimes \mathbf{1}_{l_B}, \quad \text{and} \quad T_b^{A,B} = \mathbf{1}_{l_A} \otimes T_b^B. \quad (1.61)$$

It is easy to see that the generators of the two groups will commute; indeed, this is a sign that your group is a direct product.

## E. The Simplest Groups

We are now ready to start cataloging all the Lie Groups. The simplest possible example would be a one-dimensional group, in which there will be only one generator  $T_1$ . Any single generator can always be diagonalized, so if  $T_1$  is more than one-dimensional, it is obviously block diagonal, and therefore reducible. The irreps will be one-dimensional, and are described by the value of this one-dimensional matrix, which we'll call  $T_1 = (q)$ , where  $q$  is real. The corresponding irreps are  $\Gamma^q$ , or more commonly, we'll just label them  $q$ . The representation corresponding to the coordinate  $x$  is just

$$\Gamma^q(x) = e^{iqx} \quad (1.62)$$

The trivial representation corresponds to  $q = 0$ . The tensor product of any two such irreps can be easily worked out with the help of (1.58):

$$\Gamma^{q \otimes q'} = \Gamma^{q+q'} \quad (1.63)$$

This group is already listed among our list of possible groups; indeed, it appears twice, as  $U(1)$  and as  $SO(2)$ . The set of all  $1 \times 1$  unitary matrices is the same as the complex numbers of the form  $e^{i\theta}$ . The set of all  $2 \times 2$  orthogonal matrices  $R$  with determinant +1 take the form

$$R(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \quad (1.64)$$

These are, in fact, the same group, at least mathematically. One can argue that for each of these groups, one must have  $1 = \Gamma^q(2\pi) = e^{2\pi iq}$ , so in fact  $q$  is restricted to be an integer. Note that this restriction does *not* come from the commutation relations of  $T_1$ , so in fact our claim in the previous section that the structure constants determine everything is not quite right. But this issue turns out to be slightly subtle. I will discuss it some in the next chapter, so for now we will leave it ambiguous whether there is such a restriction on  $q$ . I will subsequently always name this group  $U(1)$ .

Next consider the case of a two-dimensional Lie group. The structure constants  $f_{abc}$  are completely anti-symmetric, and since  $abc$  take on only the values 1 and 2, they must all vanish. Hence the two generators  $T_1$  and  $T_2$  will commute, and can be simultaneously diagonalized, which again implies only one-dimensional irreps. The group is  $U(1) \times U(1)$ , and its irreps are  $\Gamma^{q_1, q_2}$  or  $(q_1, q_2)$ , with

$$\Gamma^{q_1, q_2}(x_1, x_2) = e^{-iq_1 x_1} e^{-iq_2 x_2} \quad (1.65)$$

For three-dimensional groups, it is possible to have non-zero structure constants  $f_{abc}$  for the first time. Of course, it is possible that the structure constants will happen to vanish, in which case the group works out to be  $U(1) \times U(1) \times U(1)$ . If it does *not* vanish, then it must be proportional to the Levi-Civita symbol  $\varepsilon_{abc}$ ,

$$f_{abc} = f \varepsilon_{abc} \quad (1.66)$$

where the Levi-Civita symbol is given by

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = -\varepsilon_{132} = -\varepsilon_{213} = -\varepsilon_{321} = 1, \quad \text{all other components vanish} \quad (1.67)$$

Indeed, if we redefine our generators by multiplying them by a constant,  $T_a \rightarrow T_a/f$ , then we will find  $f_{abc} = \varepsilon_{abc}$ . This is the only interesting three-dimensional compact Lie group, and the subject of the next chapter.

The Levi-Civita symbol can be generalized to more dimensions; the general rule is  $\varepsilon_{ab \dots z} = +1$  if  $ab \dots z$  is an even permutation of  $12 \dots n$ ,  $\varepsilon_{ab \dots z} = -1$  if it is an odd permutation, and zero if any indices are repeated.