IV. Roots and Weights

In this chapter, we will not focus on any one particular group, but rather try to understand in general how Lie groups and their irreducible representations can be derived. We will use the groups SU(2) and SU(3) as examples whenever we can.

A. Raising and Lowering Operators

Imagine we are given an arbitrary group with a set of generators T_a, where we start knowing only the commutation relations

$$\left[T_a, T_b\right] = i f_{abc} T_c \,, \tag{5.1}$$

Just to recall, the structure constants f_{abc} can be chosen to be real and completely anti-symmetric. This will work out naturally provided our generators are "orthonormal", *i.e.*,

$$\mathrm{tr}\left(T_a^\dagger T_b\right) = \lambda \delta_{ab} \,. \tag{5.2}$$

The dagger is technically unnecessary, since our generators will always be Hermitian, but I have written it this way because this relationship will continue to work as we move to non-Hermitian generators.

The first order of business is to find as many of these generators that commute with each other as possible. If necessary, we can take linear combinations to find such generators. We will give this subset of generators the special names H_a, where a runs from 1 to k, where k is called the *rank* of the group. So we have

$$\left[H_a, H_b\right] = 0, \quad a, b = 1, \ldots, k \tag{5.3}$$

SU(2) is a rank one group, and the generator that is normally chosen is T_3. SU(3) is a rank two group, and we would use T_3 and T_8. These H's are, of course, Hermitian.

We now wish to organize the rest of the generators T_a into linear combinations that are like the raising and lowering generators of SU(2). The goal is to find new generators, which we will name something like E, which have the property that

$$\left[H_a, E\right] = r_a E \tag{5.4}$$

where r_a is a set of numbers, one for each of the H's. Now, any such generator can be written as some linear combination of the T_a's, so we write

$$E = e_i T_i \tag{5.5}$$

where e_i is just a set of complex numbers. Using (5.1) and (5.3), we see that we are trying to solve the equation

$$\left[H_a, e_i T_i\right] = i f_{aij} e_i T_j = r_a e_i T_i, \quad \text{or} \quad i f_{aji} e_j = r_a e_i \tag{5.6}$$

in other words, $i f_{aji} e_j = r_a e_i$. Now, if we recall that the adjoint representation is defined by $\left(T_a^{\mathrm{adj}}\right)_{ij} = -i f_{aij}$, we can rewrite this as

$$\left(H_a^{\text{adj}}\right)_{ij} e_j = i r_a e_i \tag{5.7}$$

This is merely an eigenvalue problem. We are trying to find eigenvectors *e* which are simultaneously eigenvectors of all the H_a^{adj} simultaneously. We know that commuting Hermitian operators can always be simultaneously diagonalized. Since these matrices are Hermitian and all commute with each other, this can be achieved, and the resulting eigenvalues r_a will all be real. Furthermore, they will span the space, *i.e.*, all of the generators can be written as linear combinations of the *E*'s.

We will label these raising/lowering operators *E* by a vector $\mathbf{r} = (r_1, r_2, \ldots, r_k)$, so they will be denoted $E_\mathbf{r}$, having the property

$$\left[H_a, E_\mathbf{r}\right] = r_a E_\mathbf{r}, \quad \text{or} \quad \left[\mathbf{H}, E_\mathbf{r}\right] = \mathbf{r} E_\mathbf{r}. \tag{5.8}$$

We will use these *only* for the generators which are not among the *H*'s. The notation implies that the raising and lowering operators are uniquely determined by their eigenvalues **r**. This is true but remains to be demonstrated. If we were to label the **H**'s in this type of designation, we would have to label them all $E_\mathbf{0}$, because they all commute with each other. If we choose the vectors e_a to be properly normalized, then the generators will still satisfy the orthonormality condition (5.3), *i.e.*,

$$\operatorname{tr}\left(H_a H_b\right) = \lambda \delta_{ab}, \quad \operatorname{tr}\left(E_\mathbf{r}^\dagger E_\mathbf{s}\right) = \lambda \delta_{\mathbf{r},\mathbf{s}}, \quad \operatorname{tr}\left(H_a E_\mathbf{r}\right) = 0. \tag{5.9}$$

For the group SU(2), the *E*'s would correspond to $E_{\pm 1} = T_\pm / \sqrt{2}$, the raising and lowering operators from before, the factor of $\sqrt{2}$ necessary to make (5.10) work out right. Similarly, for the group SU(3), the corresponding operators would be

$$E_{\pm(1,0)} = \tfrac{1}{\sqrt{2}}\left(T_1 \pm i T_2\right), \quad E_{\pm\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)} = \tfrac{1}{\sqrt{2}}\left(T_4 \pm i T_5\right), \quad E_{\pm\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right)} = \tfrac{1}{\sqrt{2}}\left(T_6 \pm i T_7\right). \tag{5.10}$$

Note that none of the $E_\mathbf{r}$'s can have $\mathbf{r} = \mathbf{0}$, since if they did, then $\left[\mathbf{H}, E_\mathbf{0}\right] = 0$, and we would then count $E_\mathbf{0}$ among the *H*'s. The list of all possible values of **r** for the $E_\mathbf{r}$'s are called the *roots* of the group. They are an intrinsic property of the group. In addition to all the non-zero **r**'s, we also include the value 0, repeated once for each of the *H*'s, so **the number of zero roots is the same as the rank of the group**. In Fig. 5-1 are sketched the roots of the group SU(3). The roots are the eigenvalues of the H_a's in the adjoint representation. In the section C, we will imagine diagonalizing the **H**'s in an arbitrary representation, and will call the resulting eigenvalues the *weights* of that representation. So the weights of the adjoint representation are the same as the roots.

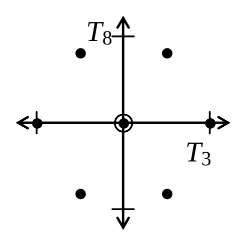


Figure 5-1: The roots of SU(3), identical with the weights of the 8 irrep of SU(3).

B. Properties of the Roots

It is easy to show that if we have found $E_{\mathbf{v}}$ satisfying (5.8), then $E_{\mathbf{v}}^\dagger$ will satisfy a similar relationship. We can easily show that

$$\left[H_a, E_{\mathbf{r}}^\dagger\right] = H_a E_{\mathbf{r}}^\dagger - E_{\mathbf{r}}^\dagger H_a = \left(E_{\mathbf{r}} H_a\right)^\dagger - \left(H_a E_{\mathbf{r}}\right)^\dagger = -\left[H_a, E_{\mathbf{r}}\right]^\dagger = -r_a E_{\mathbf{r}}^\dagger \tag{5.11}$$

so it acts like the generator $E_{-\mathbf{r}}$; *i.e.* we can choose

$$E_{-\mathbf{r}} = E_{\mathbf{r}}^\dagger \tag{5.12}$$

It follows immediately that **all the non-zero roots come in equal and opposite pairs**.

It is also not hard to show that the commutator of two of these raising/lowering operators is always another raising/lowering operator. Specifically, we have

$$\left[H_a, \left[E_{\mathbf{r}}, E_{\mathbf{s}}\right]\right] = \left[\left[H_a, E_{\mathbf{r}}\right], E_{\mathbf{s}}\right] + \left[E_{\mathbf{r}}, \left[H_a, E_{\mathbf{s}}\right]\right] = v_a\left[E_{\mathbf{r}}, E_{\mathbf{s}}\right] + u_a\left[E_{\mathbf{r}}, E_{\mathbf{s}}\right] \tag{5.13}$$

where the first identity is a variant on the Jacobi identity, but can easily be proven directly. In summary,

$$\left[\mathbf{H}, \left[E_{\mathbf{r}}, E_{\mathbf{s}}\right]\right] = \left(\mathbf{r} + \mathbf{s}\right)\left[E_{\mathbf{r}}, E_{\mathbf{s}}\right] \quad \text{or} \quad \left[E_{\mathbf{r}}, E_{\mathbf{s}}\right] \propto E_{\mathbf{r}+\mathbf{s}} \tag{5.14}$$

If there *is* no generator $E_{\mathbf{r}+\mathbf{s}}$, then we must have $\left[E_{\mathbf{r}}, E_{\mathbf{s}}\right] = 0$. Also, if $\mathbf{r} = -\mathbf{s}$, the commutator must commute with **H**, which means that it must be a linear combination of the *H*'s. Taking advantage of (5.12), we therefore have

$$\left[E_{\mathbf{r}}, E_{\mathbf{r}}^\dagger\right] = \mathbf{x} \cdot \mathbf{H} \tag{5.15}$$

We can work out the value of **x** with the help of (5.9) and the cyclic properties of the trace:

$$\begin{aligned} \operatorname{tr}\left(\mathbf{r} \cdot \mathbf{H} H_a\right) &= \operatorname{tr}\left(\left[E_{\mathbf{r}}, E_{\mathbf{r}}^\dagger\right] H_a\right) = \operatorname{tr}\left(E_{\mathbf{r}} E_{\mathbf{r}}^\dagger H_a - E_{\mathbf{r}}^\dagger E_{\mathbf{r}} H_a\right) = \operatorname{tr}\left(H_a E_{\mathbf{r}} E_{\mathbf{r}}^\dagger - E_{\mathbf{r}} H_a E_{\mathbf{r}}^\dagger\right), \\ \lambda x_a &= \operatorname{tr}\left(\left[H_a, E_{\mathbf{r}}\right] E_{\mathbf{r}}^\dagger\right) = r_a \operatorname{tr}\left(E_{\mathbf{r}} E_{\mathbf{r}}^\dagger\right) = \lambda r_a . \end{aligned} \tag{5.16}$$

So **r** = **x**, and we conclude

$$\left[E_{\mathbf{r}}, E_{\mathbf{r}}^\dagger\right] = \mathbf{r} \cdot \mathbf{H} \tag{5.17}$$

We have yet to address the issue of whether there might be more than one *E* with the same eigenvalues **r**. Suppose (using slightly inconsistent notation) we have $\mathbf{r} = \mathbf{s}$ but $E_{\mathbf{r}} \neq E_{\mathbf{s}}$. Following reasoning as before, we can write $\left[E_{\mathbf{s}}, E_{\mathbf{r}}^\dagger\right] = \mathbf{x} \cdot \mathbf{H}$. We can then run through exactly the same computations as we did in equation (5.16) and will conclude that $\lambda x_a = r_a \operatorname{tr}\left(E_{\mathbf{s}} E_{\mathbf{r}}^\dagger\right) = 0$, so we must actually have

$$\left[E_{\mathbf{r}}, E_{s}^\dagger\right] = 0 \quad \text{if} \quad \mathbf{r} = \mathbf{s} \quad \text{and} \quad E_{\mathbf{r}} \neq E_{\mathbf{s}} \tag{5.18}$$

Hence we can commute this combination wherever we want to. Now, consider the trace of $\left[E_{\mathbf{r}},E_{\mathbf{s}}\right]$ times its Hermitian conjugate, which is

$$\begin{aligned}\operatorname{tr}\left\{\left[E_{\mathbf{r}},E_{\mathbf{s}}\right]\left[E_{\mathbf{r}},E_{\mathbf{s}}\right]^{\dagger}\right\}&=-\operatorname{tr}\left\{\left(E_{\mathbf{r}}E_{\mathbf{s}}-E_{\mathbf{s}}E_{\mathbf{r}}\right)\left(E_{\mathbf{r}}^{\dagger}E_{\mathbf{s}}^{\dagger}-E_{\mathbf{s}}^{\dagger}E_{\mathbf{r}}^{\dagger}\right)\right\}\\&=-\operatorname{tr}\left\{E_{\mathbf{r}}E_{\mathbf{r}}^{\dagger}E_{\mathbf{s}}E_{\mathbf{s}}^{\dagger}-E_{\mathbf{r}}E_{\mathbf{r}}^{\dagger}E_{\mathbf{s}}^{\dagger}E_{\mathbf{s}}-E_{\mathbf{r}}^{\dagger}E_{\mathbf{r}}E_{\mathbf{s}}E_{\mathbf{s}}^{\dagger}+E_{\mathbf{r}}^{\dagger}E_{\mathbf{r}}E_{\mathbf{s}}^{\dagger}E_{\mathbf{s}}\right\}\qquad(5.19)\\&=-\operatorname{tr}\left\{\left[E_{\mathbf{r}},E_{\mathbf{r}}^{\dagger}\right]\left[E_{\mathbf{s}},E_{\mathbf{s}}^{\dagger}\right]\right\}=-\operatorname{tr}\left\{\left(\mathbf{r}\cdot\mathbf{H}\right)\left(\mathbf{s}\cdot\mathbf{H}\right)\right\}=-\lambda\mathbf{r}\cdot\mathbf{s}\end{aligned}$$

But now we have a problem. The trace of a matrix times its Hermitian conjugate is never negative. But $\lambda\left(\mathbf{r}\cdot\mathbf{s}\right)=\lambda\mathbf{r}^2>0$. Thus, we have a contradiction, implying our assumption that there are two E's with the same eigenvalue is false. **All the non-zero roots are singular**.

In general, do the roots span the space; that is, can any vector be written as a linear combinations of roots? Suppose not. Then there is some direction **x** that is perpendicular to every root **v**. Then consider the commutator

$$\left[\mathbf{x}\cdot\mathbf{H},E_{\mathbf{r}}\right]=x_a\left[H_a,E_{\mathbf{r}}\right]=x_a r_a=\mathbf{x}\cdot\mathbf{r}=0\qquad(5.20)$$

Since $\mathbf{x}\cdot\mathbf{H}$ must also commute with all the H's, it follows that $\mathbf{x}\cdot\mathbf{H}$ is a generator that commutes with all the other generators. As we discussed in chapter 1, this means that this generates a U(1) factor. **The roots span the space unless there are factors of *U*(1) in the group.** The group SU(3) is not such an example, since the roots in Fig. 5-1 clearly span the two-dimensional space.

C. Roots and Weights

We have been describing $E_{\mathbf{r}}$ as raising and lowering operators, and for good reason. Consider *any* representation of a group, and let $\left|\mathbf{w}\right\rangle$ be an eigenstate of all the H's simultaneously, so that

$$H\left|\mathbf{w}\right\rangle=\mathbf{w}\left|\mathbf{w}\right\rangle\qquad(5.21)$$

Then we call **w** a *weight* of this representation. Despite the notation, we are *not* assuming that the weights are singular, and often they will not be. It is then an easy matter to show that

$$\mathbf{H}E_{\mathbf{r}}\left|\mathbf{w}\right\rangle=\left(\left[\mathbf{H},E_{\mathbf{r}}\right]+E_{\mathbf{r}}\mathbf{H}\right)\left|\mathbf{w}\right\rangle=\mathbf{r}E_{\mathbf{r}}\left|\mathbf{w}\right\rangle+\mathbf{w}E_{\mathbf{r}}\left|\mathbf{w}\right\rangle=\left(\mathbf{w}+\mathbf{r}\right)E_{\mathbf{r}}\left|\mathbf{w}\right\rangle\qquad(5.22)$$

Hence this new state is *also* an eigenstate of all of the H's, so we logically write

$$E_{\mathbf{r}}\left|\mathbf{w}\right\rangle=N_{\mathbf{w}}\left|\mathbf{w}+\mathbf{r}\right\rangle\qquad(5.23)$$

This is starting to look just like the way we worked out irreps for SU(2), and it is. Pick any particular root **r**, then it is an easy matter to show that

$$\left[\mathbf{H},E_{-\mathbf{r}}E_{\mathbf{r}}\right]=E_{-\mathbf{r}}\left[\mathbf{H},E_{\mathbf{r}}\right]+\left[\mathbf{H},E_{-\mathbf{r}}\right]E_{\mathbf{r}}=\mathbf{r}E_{-\mathbf{r}}E_{\mathbf{r}}-\mathbf{r}E_{-\mathbf{r}}E_{\mathbf{r}}=0\qquad(5.24)$$

It follows that we can simultaneously diagonalize all the **H**'s and also the Hermitian operator $E_{-\mathbf{r}}E_{\mathbf{r}} = E_{\mathbf{r}}^{\dagger}E_{\mathbf{r}}$. With this fact, it isn't hard to show that we will have

$$E_{\mathbf{r}}^{\dagger}\left|\mathbf{w}\right\rangle = N_{\mathbf{w}-\mathbf{r}}^{*}\left|\mathbf{w}-\mathbf{r}\right\rangle \tag{5.25}$$

Start from any weight $\left|\mathbf{w}\right\rangle$, and raise repeatedly with $E_{\mathbf{r}}$ to produce the states $\left|\mathbf{w}+\mathbf{r}\right\rangle$, $\left|\mathbf{w}+2\mathbf{r}\right\rangle$, etc. Now lower repeatedly with $E_{-\mathbf{r}}$ to produce $\left|\mathbf{w}-\mathbf{r}\right\rangle$, $\left|\mathbf{w}-2\mathbf{r}\right\rangle$, etc. We will end up with a set of weights $\left|\mathbf{w}+n\mathbf{r}\right\rangle$, with n running from $-q$ to $+p$, where q is the number of times you can lower, and p the number of times you can raise. The fact that we can't raise past $n = p$ and can't lower past $n = -q$ tells us, from (5.23) and (5.25) that

$$N_{\mathbf{w}+p\mathbf{r}} = N_{\mathbf{w}-(q+1)\mathbf{r}} = 0 \tag{5.26}$$

We can get a nice formula for the normalizations $N_{\mathbf{w}}$ as follows. Using (5.17), we see that

$$\begin{aligned}
\left\langle \mathbf{w}+n\mathbf{r}\right|\left[E_{\mathbf{r}},E_{\mathbf{r}}^{\dagger}\right]\left|\mathbf{w}+n\mathbf{r}\right\rangle &= \left\langle \mathbf{w}+n\mathbf{r}\right|\mathbf{r}\cdot\mathbf{H}\left|\mathbf{w}+n\mathbf{r}\right\rangle,\\
\left\langle \mathbf{w}+n\mathbf{r}\right|\left(E_{\mathbf{r}}E_{\mathbf{r}}^{\dagger}-E_{\mathbf{r}}^{\dagger}E_{\mathbf{r}}\right)\left|\mathbf{w}+n\mathbf{r}\right\rangle &= \mathbf{r}\cdot\left(\mathbf{w}+n\mathbf{r}\right),\\
\left|N_{\mathbf{w}+(n-1)\mathbf{r}}\right|^{2}-\left|N_{\mathbf{w}+n\mathbf{r}}\right|^{2} &= \mathbf{r}\cdot\mathbf{w}+n\mathbf{r}^{2}
\end{aligned} \tag{5.27}$$

Now take equation (5.27) and sum it over all n:

$$\begin{aligned}
\sum_{n=-q}^{p}\left(\mathbf{r}\cdot\mathbf{w}+n\mathbf{r}^{2}\right) &= \sum_{n=-q}^{p}\left|N_{\mathbf{w}+(n-1)\mathbf{r}}\right|^{2}-\sum_{n=-q}^{p}\left|N_{\mathbf{w}+n\mathbf{r}}\right|^{2},\\
\left(p+q+1\right)\mathbf{r}\cdot\mathbf{w}+\left(\frac{p^{2}+p}{2}-\frac{q^{2}-q}{2}\right)\mathbf{r}^{2} &= \sum_{n=-q-1}^{p-1}\left|N_{\mathbf{w}+n\mathbf{r}}\right|^{2}-\sum_{n=-q}^{p}\left|N_{\mathbf{w}+n\mathbf{r}}\right|^{2},\\
\left(p+q+1\right)\left[\mathbf{r}\cdot\mathbf{w}+\tfrac{1}{2}\left(p-q\right)\mathbf{r}^{2}\right] &= \left|N_{\mathbf{w}-(q+1)\mathbf{r}}\right|^{2}-\left|N_{\mathbf{w}+p\mathbf{r}}\right|^{2}=0,\\
\mathbf{r}\cdot\mathbf{w}+\tfrac{1}{2}\left(p-q\right)\mathbf{r}^{2} &= 0,\\
q-p &= \frac{2\mathbf{r}\cdot\mathbf{w}}{\mathbf{r}^{2}}
\end{aligned} \tag{5.28}$$

where we recall that q is the number of times we can lower and p is the number of times we can raise. This simple rule will help us work out the irreps for any group. The first, and most important fact we deduce from this is that since q and p are integers, $2\mathbf{r}\cdot\mathbf{w}/\mathbf{r}^{2}$ is an integer. This would be akin to saying, in the group SU(2), that $2T_3$ always has integer eigenvalues. The other fact I would like to note is that if $\mathbf{r}\cdot\mathbf{w}<0$, then we can definitely raise at least once, so $E_{\mathbf{r}}\left|\mathbf{w}\right\rangle \neq 0$.

To explain the significance of (5.28), consider the group SU(3), for which there are three pairs of roots. If we pick one of them, then (5.28) tells us that the weights will always end up symmetrically arranged in the **r**-direction. For example, if $\mathbf{r}=\left(1,0\right)$, then the weight diagrams are always symmetric under reflection across the y-axis.

D. Roots and Roots

Recall in section A that we showed that the weights of the adjoint representation are the roots. It follows that equation (5.28) must be valid if we let **w** be a root instead. We therefore have

$$q - p = \frac{2\mathbf{r}\cdot\mathbf{s}}{\mathbf{r}^2} \tag{5.29}$$

where **r** and **s** are both roots. This must be an integer, which tells you how many times you can add **r** to **s** to produce new roots, so that $\mathbf{s} - q\mathbf{r}, s - (q-1)\mathbf{r}, \ldots, \mathbf{s} + p\mathbf{r}$ are all roots.

We will be needing a concept of "positive" to describe both the roots and the weights. How positive is defined isn't really important, but for definiteness, we will define it as follows: a root is *positive* if its first non-zero component is positive, so for example, if $r_1 > 0$, or if $r_1 = 0$ and $r_2 > 0$, etc. A root **r** is *negative* if $-\mathbf{r}$ is positive. With these definitions, it is easy to see that any vector is either positive, negative, or zero, and also that the sum of two positive roots is always positive. For SU(3), the positive roots are $\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$ and $(1,0)$.

A *simple* root is a positive root that cannot be written as the sum of two other positive roots. These roots are especially important. Suppose **r** and **s** are two roots with corresponding generators $E_\mathbf{r}$ and $E_\mathbf{s}$, then we already know that $\left[E_\mathbf{r}, E_\mathbf{s}\right] \propto E_{\mathbf{r}+\mathbf{s}}$. Hence it won't really be necessary to specify $E_{\mathbf{r}+\mathbf{s}}$, since we can obtain it by commuting $E_\mathbf{r}$ and $E_\mathbf{s}$. Hence, in general, if we can find the generators for the simple roots, we can find all of the generators. For the group SU(3), there will only be two simple roots, $\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$.

It is pretty easy to see that we can write all the positive roots as combinations of the simple roots. Suppose **r** and **s** are two distinct simple roots, and let's assume without loss of generality that $\mathbf{r} > \mathbf{s}$. Then $\mathbf{r} - \mathbf{s}$ is positive, but it is not a root (if it were, then we could write $\mathbf{r} = \mathbf{s} + (\mathbf{r} - \mathbf{s})$, so **r** would not be simple), and since the roots come in equal and opposite pairs, neither is $\mathbf{s} - \mathbf{r}$. Therefore we can never subtract simple roots, we can only add them. In (5.29), this means that if **r** and **s** are simple roots, we must have $q = 0$, so it immediately follows that $\mathbf{r}\cdot\mathbf{s} \le 0$.

Let **r** and **s** be two simple roots. The combination (5.29), $2\mathbf{r}\cdot\mathbf{s}/\mathbf{r}^2$, must be an integer. If we swap the role of **r** and **s**, then it is also true that $2\mathbf{r}\cdot\mathbf{s}/\mathbf{s}^2$ is an integer as well. Multiplying these two equations, we conclude that

$$\frac{2\mathbf{r}\cdot\mathbf{s}}{\mathbf{r}^2}\cdot\frac{2\mathbf{r}\cdot\mathbf{s}}{\mathbf{s}^2} = 4\cos^2\theta \tag{5.30}$$

is an integer, where θ is the angle between **r** and **s**. Together with the information that $\mathbf{r}\cdot\mathbf{s} \le 0$, and that **r** and **s**, since they are both positive, are not in the same direction, severely restricts the possible angles θ. It is also possible to figure out the possible ratios of the relative size of the two roots. Table 5-1 above lists all the possibilities.

Table 5-1: A list of all possible angles and relative sizes for any two simple roots **r** and **s**.

θ	90°	120°	135°	150°
$\lvert\mathbf{r}\rvert/\lvert\mathbf{s}\rvert$	any	1	$\sqrt{2}, \frac{1}{\sqrt{2}}$	$\sqrt{3}, \frac{1}{\sqrt{3}}$

E. Dynkin Diagrams

I would like to point out that it is not the actual values of the roots that matter, but rather their geometry, their relative lengths and angles. The lengths of all the roots can be changed arbitrarily simply by multiplying all our generators by a common constant. Furthermore, the whole system of roots can be rotated arbitrarily merely by choosing different combinations of the mutually commuting generators H_a. It is only the angles and relative sizes of the roots that actually matter.

We now develop a diagrammatic method of describing the simple roots called a *Dynkin diagram*. This technique does *not* include any U(1) factors in the group, which must be accounted for separately, because U(1) factors do not involve any raising and lowering operators, and hence have no non-zero roots associated with them. The technique is that we will draw a circle for each simple root. Depending on which of the four cases in Table 5-1 apply, we will join each pair of simple roots by zero, one, two, or three lines. If the roots are not joined, we have no knowledge of their relative length. If they are joined by a single line, they are the same length. If they are joined by a double or triple line, we will also shade the root that is longer (by $\sqrt{2}$ or $\sqrt{3}$ respectively). The four different cases are illustrated in Fig. 5-2 at right.

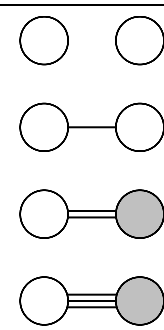


Figure 5-2: The four ways that two simple roots can be related in a Dynkin diagram. From top to bottom, the angles are 90°, 120°, 135°, and 150°. In the bottom two diagrams, the longer root is shaded.

The group SU(2) has the simplest Dynkin diagram, with one simple root, so it is designated by a single circle. The group SU(3) has two simple roots, and is connected by a single line, designating the 120 degree angle between the roots, as illustrated in Fig. 5-3.

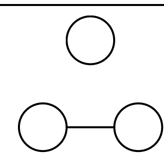


Figure 5-3: The Dynkin diagram for SU(2) (top) and SU(3) (bottom).

Given a Dynkin diagram, it is possible to generate all of the roots of the group, with the help of (5.29). For example, let's find all the roots for the group G_2, a rank two group (which implies two simple roots), whose Dynkin diagram is given in Fig. 5-4 at right. According to the Dynkin diagram, this group will have two simple roots, which I will denote **r** and **s**, which are 150° apart, and one of them is $\sqrt{3}$ times longer than the other. For example, we can choose

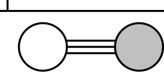


Figure 5-4: The Dynkin diagram for G_2.

$$\mathbf{r} = (0,1), \quad \mathbf{s} = \left(\tfrac{\sqrt{3}}{2}, -\tfrac{3}{2}\right) \tag{5.31}$$

Now, all the positive roots are sums of these roots. This suggests adding **r** to **s** or vice versa. How many times can we add **r** to **s**? Using equation (5.29), the answer is

$$q - p = \frac{2\mathbf{r}\cdot\mathbf{s}}{\mathbf{r}^2} = -3 \tag{5.32}$$

Since we can't subtract a simple root from a simple root, $q = 0$, so $p = 3$. Hence we can add **r** to **s** three times, which yields three new roots:

$$\{\mathbf{s}+\mathbf{r},\mathbf{s}+2\mathbf{r},\mathbf{s}+3\mathbf{r}\}=\left\{\left(\tfrac{\sqrt{3}}{2},-\tfrac{1}{2}\right),\left(\tfrac{\sqrt{3}}{2},\tfrac{1}{2}\right),\left(\tfrac{\sqrt{3}}{2},\tfrac{3}{2}\right)\right\} \tag{5.33}$$

A similar argument can be used to show that since $2\mathbf{s}\cdot\mathbf{r}/\mathbf{s}^2=-1$, we can add only one **s** to **r**, so $\mathbf{r}+\mathbf{s}$ is a root but $\mathbf{r}+2\mathbf{s}$ is not.

We aren't generally done at this point. We can try adding **s** again to any of the roots given above. We already know we can't add it to **s** + **r**, but for the others, we have

$$\begin{aligned}\mathbf{s}+2\mathbf{r}: &\quad q-p=2(\mathbf{s}+2\mathbf{r})\cdot\mathbf{s}/\mathbf{s}^2=0,\\ \mathbf{s}+3\mathbf{r}: &\quad q-p=2(\mathbf{s}+3\mathbf{r})\cdot\mathbf{s}/\mathbf{s}^2=-1.\end{aligned} \tag{5.34}$$

In each case, we can't subtract **s**, so this tells us we can add **s** to the first one (just once) but not the second. Hence $2\mathbf{s}+3\mathbf{r}=\left(\sqrt{3},0\right)$ is also a root, but $3\mathbf{s}+3\mathbf{r}$ is not. Can we get any more? We can't add **s** to $2\mathbf{s}+3\mathbf{r}$, and since $(2\mathbf{s}+3\mathbf{r})\cdot\mathbf{s}=0$, we can't add **r** either. So we have found all the positive roots!

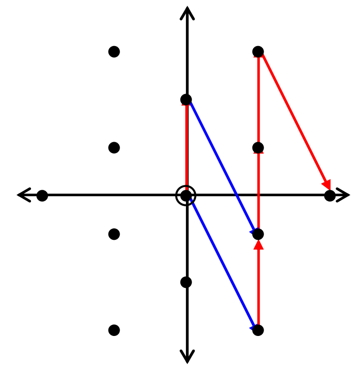


Figure 5-5: Method for finding the roots of G_2. The roots are found by repeatedly adding the roots **r** (red arrows) and **s** (blue arrows). The negative roots are then just the negatives of the positive roots.

The negative roots are negatives of the positive roots. And since we have two simple roots, it is a rank two group, and therefore we have two zero roots. The method of finding all the roots of G_2 is illustrated in Fig. 5-5.

Note that if two simple roots are not connected by a line, they are at 90 degrees with each other, and therefore $\mathbf{r}\cdot\mathbf{s}=0$, which implies that there is no $E_{\mathbf{r}+\mathbf{s}}$ and therefore $[E_\mathbf{r},E_\mathbf{s}]=0$. Indeed, if you have a Dynkin diagram with two or more piecess that are completely disconnected, then any combination of the raising generators from one set and the raising generators from the other set will commute with each other. Hence the resulting group will always produce two subsets of generators that completely commute with each other. The result, it is not hard to see, is that your group will consist of sets of generators that completely commute with each other. As discussed in chapter 1, this means that the group is a direct product. For example, the Dynkin diagram in Fig. 5-6 corresponds to the group $SU(3)\times SU(2)\times G_2$, because it consists of these three disconnected Dynkin diagrams.

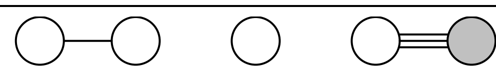


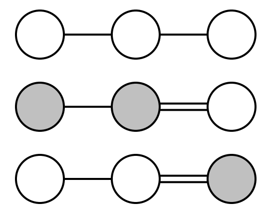
Figure 5-6: Disconnected Dynkin diagrams describe direct product groups, in this case, the group $SU(3)\times SU(2)\times G_2$.

We turn our attention now to attempting to find all the disconnected Dynkin diagrams, and then we will build all Lie groups by taking direct products of the list of groups we find. Before we begin, however, let us not forget the group $U(1)$, which is not described by a Dynkin diagram, since it has no non-zero roots.

F. The Classification Theorem

The goal of this section is to find all possible connected Dynkin diagrams, using only geometry. We have already found all possible Dynkin diagrams with two simple roots, they are illustrated in Fig. 5-2 (not counting the first one). We will use only our knowledge of the angles and lengths involved. We will also use one other fact: since the simple roots span the space, no non-trivial linear combination of simple roots can ever yield zero.

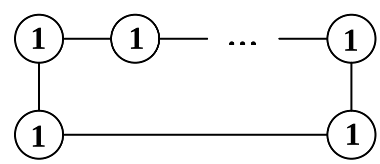
Let's start by tackling the problem of three simple roots. If three vectors lie in a plane, their angles never add up to more than 360 degrees. But since our simple roots are linearly independent, they *don't* lie in a plane. In 3D, it isn't hard to convince yourself that the three angles must total less than 360 degrees. The only connected diagrams with this small a total are the three illustrated at right. Note that, in fact, there are no triple lines; the only diagram with a triple line is G_2. Any system with more simple roots than this will contain these as a part of the diagram.



***Rule 1:* The only Dynkin diagram with a triple line is G_2.**

Now, as we go beyond three dimensions, our intuition will get less reliable, and we'll have to do more math. I will exclude a variety of diagrams, and then examine all possible remaining diagrams. We will do proofs that rely on the simple fact that any linear combination of simple roots cannot equal zero. Basically, in every case, I will find a linear combination of the roots and square it and prove that the result is zero. This will require that I find dot products like $2\mathbf{r}\cdot\mathbf{s}/\mathbf{r}^2$. If two roots are not joined by a line, this is zero; if joined by a single line, it is -1. If joined by a double line, it will be -1 if $\mathbf{r}$ is the longer (shaded) root, and -2 if it is the shorter (unshaded) root.

Let me give a quick example of a diagram I can prove will not work. Consider the circular diagram at right, with N simple roots arranged in a circle. The numbers are the numbers I will multiply each root by. In other words, I want to consider the combination



$$\mathbf{v} = \mathbf{r}_1 + \mathbf{r}_2 + \cdots + \mathbf{r}_N \qquad (5.35)$$

I now square out $\mathbf{v}^2$, keeping in mind that the only terms that won't vanish will be the dot products of each root with themselves, together with the dot products of connected roots. In other words,

$$\mathbf{v}^2 = \mathbf{r}_1^2 + \mathbf{r}_2^2 + \cdots + \mathbf{r}_N^2 + 2(\mathbf{r}_1\cdot\mathbf{r}_2) + 2(\mathbf{r}_2\cdot\mathbf{r}_3) + \cdots 2(\mathbf{r}_N\cdot\mathbf{r}_1) \qquad (5.36)$$

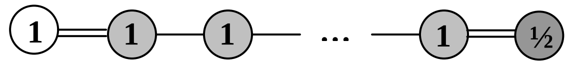
Now, all the roots are connected with single lines, so they are all the same length. If we call this common length r, then we know that each of the squared terms is r^2 and each of the dot products is $2\mathbf{r}_1\cdot\mathbf{r}_2 = \left[2\mathbf{r}_1\cdot\mathbf{r}_2/\mathbf{r}_1^2\right]\mathbf{r}_1^2 = -r^2$. Hence we have

$$\mathbf{v}^2 = r^2(1+1+\cdots+1-1-1-\cdots-1) = 0 \qquad (5.37)$$

But the sum of positive roots must be positive, so we have a contradiction. Hence there can be no cycle of roots connected with single lines.

Rule 2: ***No loops.***

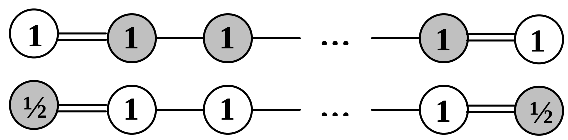
Let's do a second proof, so we understand what is going on. How about the diagram at right, with two double lines, such that most of the roots (call them $\mathbf{r}_i$) are the same length, but one of them is shorter (call is **s**) and one of them is longer (call it **l**). Consider the combination



$$
\begin{aligned}
\mathbf{v}^2 &= \left(\mathbf{s}+\mathbf{r}_1+\mathbf{r}_2+\cdots+\mathbf{r}_N+\tfrac{1}{2}\mathbf{l}\right)^2 \\
&= \mathbf{s}^2+\mathbf{r}_1^2+\mathbf{r}_2^2+\cdots+\mathbf{r}_N^2+\tfrac{1}{4}\mathbf{l}^2+2\mathbf{s}\cdot\mathbf{r}+2\mathbf{r}_1\cdot\mathbf{r}_2+2\mathbf{r}_2\cdot\mathbf{r}_3+\cdots+2\mathbf{r}_{N-1}\cdot\mathbf{r}_N+\mathbf{r}_N\cdot\mathbf{l} \\
&= r^2\left[\tfrac{1}{2}+1+1+\cdots+1+\tfrac{1}{4}\cdot 2+2\mathbf{s}\cdot\mathbf{r}/r^2+2\mathbf{r}_1\cdot\mathbf{r}_2/r^2+\cdots+2\mathbf{r}_{N-1}\cdot\mathbf{r}_N/r^2+\mathbf{r}_N\cdot\mathbf{l}/r^2\right],
\end{aligned}
$$

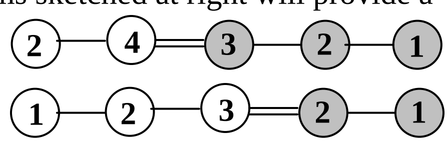
$$\mathbf{v}^2 = r^2\left[\tfrac{1}{2}+N+\tfrac{1}{2}-1-(N-1)-1\right]=0 \tag{5.38}$$

Similar proofs can be provided for the other two ways of including double lines. The proofs are sketched at right. The inescapable conclusion is that we can have, at most, one double line anywhere in a connected diagram.



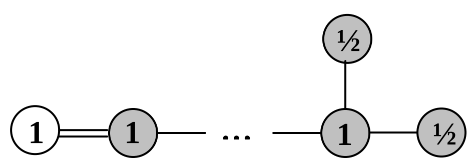
Rule 3: ***At most one double line***

Indeed, it turns out that if we have a double line anywhere, we can't have very long chains coming off of it either. The two diagrams sketched at right will provide a proof that if we have another root attached at both ends, we can't add any more roots. If there is a root attached at both ends, that's all we can have.

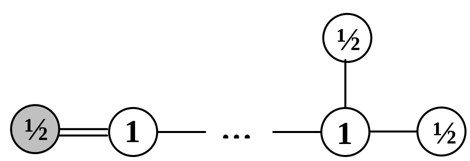


Rule 4: ***If you have chains on both sides of a double line, the chains can't be longer than one root each.***

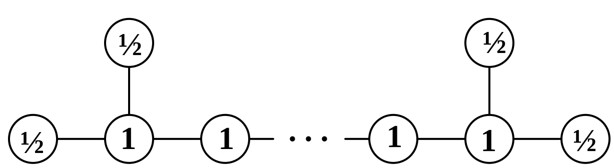
You also can't have any sort of branching going on with any system of roots involving a double line. Suppose you have a double line connected to a branch, directly or indirectly. One of the diagrams at right must be what is going on. In each case, you discover that you get zero.



Rule 5: ***No double lines and branches in the same diagram.***

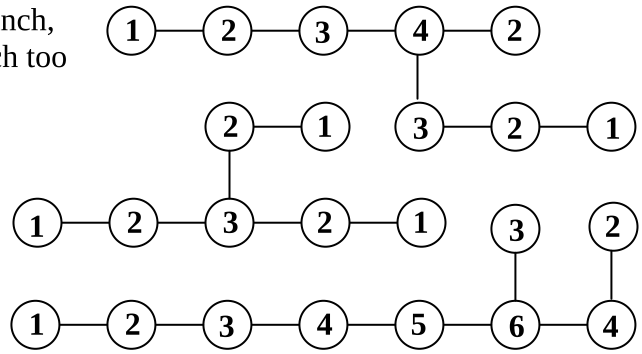


We have now exhausted our options for any diagram containing a double line. For single lines, the only case still remaining is various types of branching, but even this can't get too complicated. The diagram at right shows you can't have more than one branch, for example.



Rule 6: ***At most one branch.***

Even when you have a single branch, you can't have all three legs of the branch too long, as illustrated at right.



Rule 7: ***The three ends of a branch can't be as long as those illustrated at right.***

These seven rules allow us to classify all Dynkin diagrams The set of all allowed Dynkin diagrams are listed in Fig. 5-7, at right. They are labeled by their rank, which is the same as the number of simple roots. The lettering is that given by Cartan, who first worked these out. The A_N's work out to be nothing more than the groups SU(N+1). The B_N's and D_N's work out to be the groups SO(2N+1) and SO(2N) respectively. The C_N's are the Sp(N)'s. The five remaining groups, called the *exceptional* groups, have no other names with which I am familiar. The most general Lie group is the direct product of any combination of these groups and also the group U(1).

If the class weren't ending, I would spend lots more time discussing these various cases, especially the very interesting SO(N) groups, but alas, the semester is over, and if you want to learn more you'll just have to become a particle physicist, or consult Georgi for more information.

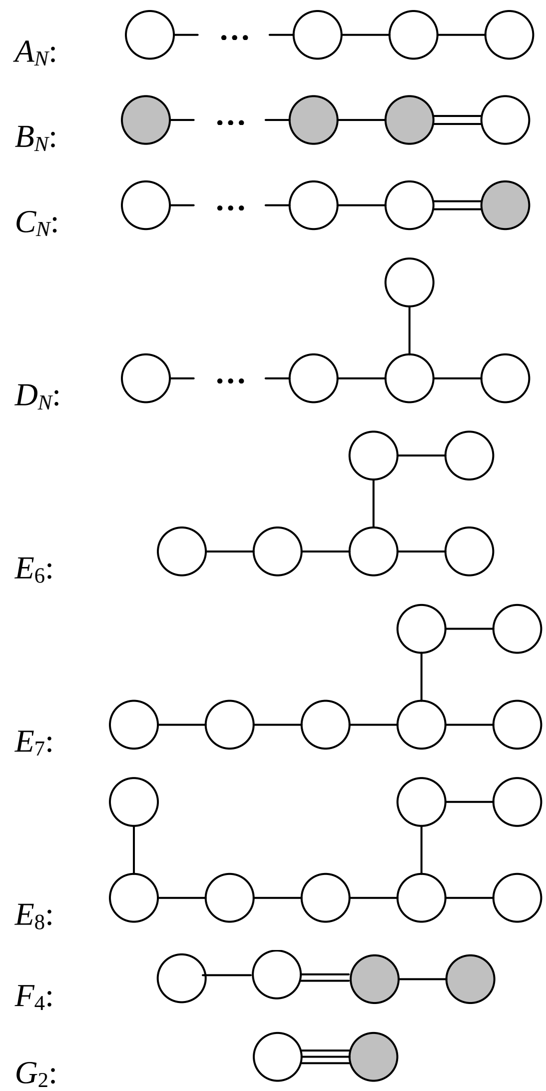


Figure 5-7: The set of all connected Dynkin diagrams.