Particle Physics, Fall 2012 **Solutions to Final Exam** December 11, 2012

Part I: Short Answer [100 points]
For each of the following, give a short answer (2-3 sentences, or a formula). [5 points each]

1. **[This one might be hard for Sam]: What is your name? ___________________________**

2. **The cross section ratio $\sigma\left(e^+e^- \to \text{hadrons}\right)\big/\sigma\left(e^+e^- \to \mu^+\mu^-\right)$ is usually about flat, but occasionally undergoes steps up, where the "constant" value increases to a new, higher constant. What is causing those steps up?**

The cross-section is proportional to the sum of all the quark charges less than the appropriate energy. As you increase past the successive quark masses, the ratio increases.

3. **Suppose you had calculate the amplitude for some quantum mechanical process involving photons, say $X \to Y\gamma$, and it takes the form $i\mathcal{M} = f^\alpha \varepsilon_\alpha^*$, where f^α is something complicated, and ε_α is the polarization of the photon. What could you do to check if your answer is gauge invariant?**

Gauge invariance can be checked by replacing the polarization with its corresponding momentum and see if the result vanishes. In other words, you check if $f^\alpha q_\alpha = 0$.

4. **Isospin is an abbreviation of "isotopic spin." What sort of isotopes can be related by isospin? For example, can it tell you how the energy levels of ^{4}He nuclei relate to energy levels of ^{238}U nuclei?**

Isospin converts protons to neutrons. Hence it can relate different isotopes with the same mass number *A*, but it can't relate nuclei with different *A*. In particular ^{4}He and ^{238}U can't be related.

5. **Which of the isospin or SU(3) flavor is a better approximation to reality? Is either of them a perfect symmetry? Give an argument for why they are imperfect, or not.**

Isospin is a better symmetry than SU(3). We can tell neither is perfect, since if it were, π^+, π^0, and K^0 would all have the same mass, which they don't. In fact, the π^+ and π^0 are close in mass (suggesting isospin is good but not perfect) while the K^0 is rather different.

6. **We used SU(3) twice in this class, and called them SU(3)$_F$ and SU(3)$_C$. Suppose I had a red up quark. What sort or particle might SU(3)$_F$ relate it to? What sort of particle might SU(3)$_C$ relate it to?**

SU(3)$_F$ relates quarks of different flavors. So SU(3)$_F$ can relate a red up quark into a red down or red strange quark. SU(3)$_C$ relates quarks of different color, so it can relate a red up quark to a blue or green up quark.

7. **The $\left|\Sigma^+\right\rangle$ is a strangeness -1 baryon with charge +1. What is its quark content? Assume it contains only some subset of the three lightest quarks.**

Since it has strangeness -1, it must have one strange quark with charge $-\frac{1}{3}$. Since the total charge is +1, the other two quarks must have charge $+\frac{2}{3}$ to make the charge work out, so they are up quarks. So $\left|\Sigma^+\right\rangle = \left|uus\right\rangle$.

8. **Free quarks have never been discovered. What happens, qualitatively, if you try to separate the quarks, say, in a meson, like $\left|\pi^+\right\rangle = \left|u\bar{d}\right\rangle$?**

If you try to separate them, a flux tube will form between them. As you move them farther apart, the flux tube will have so much energy that it can create a new quark anti-quark pair, and we will just end up with two mesons.

9. **A $\left|\Xi^{*0}\right\rangle$ spin $\frac{3}{2}$ baryon with spin $S_z = +\frac{3}{2}$ would have its spin and quark content described as $\left|\Xi^{*0}, +\frac{3}{2}\right\rangle = \frac{1}{\sqrt{3}}\left(\left|u\uparrow, s\uparrow, s\uparrow\right\rangle + \left|s\uparrow, u\uparrow, s\uparrow\right\rangle + \left|s\uparrow, s\uparrow, u\uparrow\right\rangle\right)$. But this appears to be completely symmetric, yet it involves only fermions. How is this apparent discrepancy resolved?**

In addition to the spin and flavor, quarks have color. Though the spin and flavor part is symmetric, the color part, which is not written above, is completely anti-symmetric. So it's okay.

10. **How many colors are there for the electron, electron neutrino, up quark, down quark, gluon, and photon?**

The electron, electron neutrino, and photon are colorless, so there is just one color (white). There are three colors for the up and down quark. There are eight colors for the gluons.

11. As you go to higher energies, it is known that the electromagnetic fine structure constant α gets stronger. Is this true of the strong coupling α_S as well?

No, the strong coupling constant gets weaker at higher energy, leading to what is called "asymptotic freedom."

12. The first theory of weak interactions we came up was the Fermi theory, with a coupling G_F which led to Feynman amplitudes like $i\mathcal{M} \sim G_F \left[\bar{u}\gamma^\mu (1-\gamma_5) v\right]\left[\bar{u}'\gamma_\mu (1-\gamma_5) v'\right]$. Why did we ultimately abandon this theory?

The theory is non-renormalizable, as is evident from the fact that G_F is of dimension mass to the minus two. As a consequence, the probability increases as you increase your energy, ultimately passing one.

13. Why are weak interactions weak? In other words, why are processes like $\mu^- \to e^- \bar{\nu}_e \nu_\mu$ so slow?

The interaction involves an intermediate *W* boson, which has a high mass. This leads to a large denominator in the propagator which suppresses the amplitude.

14. What is the difference between charged current and neutral current weak interactions? What particles are responsible for each of them?

Charged current interactions always involve particles changing charge by one unit, like $\mu^- \leftrightarrow \nu_\mu$ or $d \leftrightarrow u$ conversions. Neutral currents do not, they always connect fermions to other fermions of the same type $f \leftrightarrow f$. Charged currents are always mediated by *W*-bosons, and neutral current by *Z*-bosons.

15. What is the gauge group of the standard model?

The gauge group is $SU(3)_c \times SU(2)_L \times U(1)_Y$, or just $SU(3) \times SU(2) \times U(1)$.

16. **Naively, you would think there is a an "up quark" field, with both left- and right-handed parts, and a "down quark" field, with both left- and right-handed quarks. But if it weren't for symmetry breaking, these associations would be meaningless. What are the three actual fields that account for the up quark and down quark?**

The left handed up and down quarks are together in a weak doublet q_L, while the right quarks are separate fields, u_R and d_R.

17. **Because the Higgs field is a complex doublet, you would think that there would be four degrees of freedom, and hence four distinct particles. But only a single degree of freedom (the Higgs boson) has been discovered, and this is all that is expected. What happened to those other degrees of freedom?**

The other three degrees of freedom are "eaten" by the $W^{\pm}$ and Z^{0} boson, which allows then to become massive, and makes them gain a third polarization or degree of freedom.

18. **It is known experimentally that there is *CP* violation. Where in the standard model can we find a source of *CP* violation?**

It exists only in the CKM matrix, that describes how up-type quarks and down-type quarks are connected by the *W*-boson.

19. **The most common decay of the Higgs is $H \to b\bar{b}$. Yet this decay was not the (primary) one that was used at the LHC to detect the Higgs. Why is this decay difficult to distinguish?**

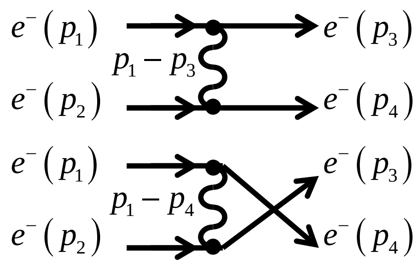
Quarks are produced in copious quantities via strong interactions, and therefore there is a massive background for this process. This makes it very difficult to pick out this signal from the noise.

20. **When we calculate cross-sections like $\nu_e e^- \to \nu_e e^-$, we sum over the final spins, average over the initial electron spin, and <u>sum</u> over the initial neutrino spin. Why is this the right thing to do?**

Neutrinos all are left-handed in the standard model. Fortunately, the couplings are zero for the wrong helicity, so summing over both spins only adds zero to the total, which is the correct thing to do.

Part II: Calculation [200 points]
Each problem has its corresponding point value marked. Solve the equations on separate paper.

21. [20] Draw the Feynman diagrams relevant for electron-electron scattering, $e^-(p_1)e^-(p_2) \to e^-(p_3)e^-(p_4)$. Write the Feynman amplitude, including the correct relative sign. Assume you are at low enough energy that only QED contributions are important.



There are two diagrams, sketched at right. Since they differ only by switching external fermion lines, there will be a relative minus sign between them. The electron has charge -1, so the coupling is $ie\gamma^\mu$, and the resulting amplitude is

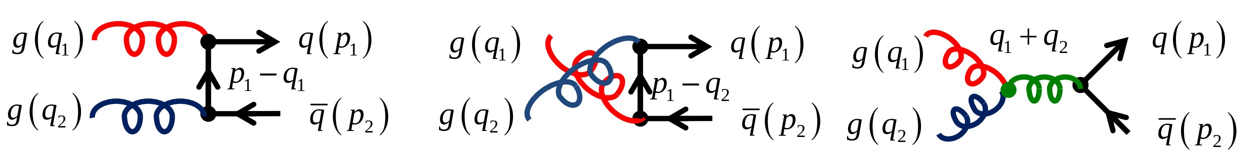
$$i\mathcal{M} = (ie)^2 \left\{ (\bar{u}_3\gamma^\mu u_1)(\bar{u}_4\gamma^\nu u_2)\frac{-ig_{\mu\nu}}{(p_1-p_3)^2} - (\bar{u}_4\gamma^\mu u_1)(\bar{u}_3\gamma^\nu u_2)\frac{-ig_{\mu\nu}}{(p_1-p_4)^2} \right\}$$
$$= ie^2 \left[\frac{(\bar{u}_3\gamma^\mu u_1)(\bar{u}_4\gamma_\mu u_2)}{(p_1-p_3)^2} - \frac{(\bar{u}_4\gamma^\mu u_1)(\bar{u}_3\gamma_\mu u_2)}{(p_1-p_4)^2} \right].$$

22. [10] In SU(3)$_F$ notation, the Ξ^0 particle is $\left|\Xi^0\right\rangle = \left|B_3^2\right\rangle$. Find the effects of all six $T_{i\to j}$ operators on $\left|\Xi^0\right\rangle$.

We recall that $T_{i\to j}$ turns *i* into *j* if it acts on a lower index, and it turns *j* into *i* and adds a minus sign if it acts on an upper index. Hence

$$T_{1\to 2}\left|B_3^2\right\rangle = -\left|B_3^1\right\rangle, \qquad T_{2\to 3}\left|B_3^2\right\rangle = 0, \qquad T_{3\to 1}\left|B_3^2\right\rangle = \left|B_1^2\right\rangle,$$
$$T_{2\to 1}\left|B_3^2\right\rangle = 0, \qquad T_{3\to 2}\left|B_3^2\right\rangle = \left|B_2^2\right\rangle - \left|B_3^3\right\rangle, \qquad T_{1\to 3}\left|B_3^2\right\rangle = 0.$$

23. [15] Draw all relevant tree-level Feynman diagrams for a pair of gluons to collide and make a quark/anti-quark pair, $g(q_1)g(q_2) \to q(p_1)\bar{q}(p_2)$. Correctly label any intermediate momenta that arise. You don't have to calculate the amplitude.



The three diagrams, together with the appropriate intermediate momenta, are drawn above.

24. [20] The three pions $\left(\pi^-,\pi^0,\pi^+\right)$ form an isospin triplet, *I* = 1, for which we have

$$I_+ = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}, \quad I_- = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}.$$

(a) Work out $I_+\left|\pi\right\rangle$ and $I_-\left|\pi\right\rangle$ for all three particles.

The isospin operators always increase or decrease the charge by exactly one unit. The factor is clearly going to be $\sqrt{2}$, so we have

$$I_+\left|\pi^-\right\rangle = \sqrt{2}\left|\pi^0\right\rangle, \quad I_+\left|\pi^0\right\rangle = \sqrt{2}\left|\pi^+\right\rangle, \quad I_+\left|\pi^+\right\rangle = 0,$$
$$I_-\left|\pi^-\right\rangle = 0, \quad I_-\left|\pi^0\right\rangle = \sqrt{2}\left|\pi^-\right\rangle, \quad I_-\left|\pi^+\right\rangle = \sqrt{2}\left|\pi^0\right\rangle.$$

(b) The ω is an isospin singlet, so $I_a\left|\omega\right\rangle = 0$. If isospin is a perfect symmetry, argue that $\left\langle \pi^+\pi^0\right|I_+\mathcal{H}\left|\omega\right\rangle = 0$.

If isospin is a perfect symmetry, then I_+ commutes with the Hamiltonian. Hence

$$\left\langle \pi^+\pi^0\right|I_+\mathcal{H}\left|\omega\right\rangle = \left\langle \pi^+\pi^0\right|\mathcal{H}I_+\left|\omega\right\rangle = 0.$$

(c) Show from part (b) that there is a simple relationship between $\left\langle \pi^+\pi^-\right|\mathcal{H}\left|\omega\right\rangle$ and $\left\langle \pi^0\pi^0\right|\mathcal{H}\left|\omega\right\rangle$.

We first note that

$$I_-\left|\pi^+\pi^0\right\rangle = \sqrt{2}\left|\pi^0\pi^0\right\rangle + \sqrt{2}\left|\pi^+\pi^-\right\rangle.$$

Taking the Hermitian conjugate of this equation, we see that

$$\left\langle \pi^+\pi^0\right|I_- = \sqrt{2}\left\langle \pi^0\pi^0\right| + \sqrt{2}\left\langle \pi^+\pi^-\right|,$$

We therefore have

$$0 = \left\langle \pi^+\pi^0\right|I_-\mathcal{H}\left|\omega\right\rangle = \sqrt{2}\left\langle \pi^0\pi^0\right|\mathcal{H}\left|\omega\right\rangle + \sqrt{2}\left\langle \pi^+\pi^-\right|\mathcal{H}\left|\omega\right\rangle,$$
$$\left\langle \pi^0\pi^0\right|\mathcal{H}\left|\omega\right\rangle = -\left\langle \pi^+\pi^-\right|\mathcal{H}\left|\omega\right\rangle.$$

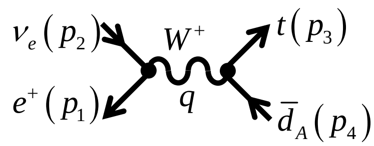
25. [25] Consider the hypothetical process for production of top quarks by colliding neutrinos with positrons,

$$e^+(p_1)\nu_e(p_2) \to t(p_3)\bar{f}(p_4)$$

where *f* represents some sort of final state fermion

(a) Which possible fermions could it be? There should be multiple correct answers. Which fermion is most likely to be produced?

By conservation of charge, the initial charge was 1+0 = 1, and the quark has charge $+\frac{2}{3}$, which means that the missing charge must be $+\frac{1}{3}$. This implies it must be the anti-particle of one of the down-type quarks, so it is $\bar{d}$, $\bar{s}$, or $\bar{b}$. The relevant Feynman diagram is sketched at right There will be a CKM matrix element involved, which will favor the bottom quark over any others in this process.



(b) Draw the relevant Feynman diagram.

(c) Write the Feynman amplitude for this process. For the moment, let *q* be the momentum of any intermediate particle that you need for the process.

The Feynman diagram is above. The amplitude can be found from the rules on the handout, so we have

$$i\mathcal{M} = \left(-\frac{ie}{2\sqrt{2}\sin\theta_W}\right)^2 V_{tA}\left[\bar{v}_1\gamma^\mu(1-\gamma_5)u_2\right]\left[\bar{u}_3\gamma^\nu(1-\gamma_5)v_4\right]\frac{i\left(-g_{\mu\nu}+q_\mu q_\nu/M_W^2\right)}{q^2-M_W^2}$$

(d) Write *q* in two ways in terms of the p_i's.

From straightforward conservation of momentum, we have $q = p_1 + p_2 = p_3 + p_4$.

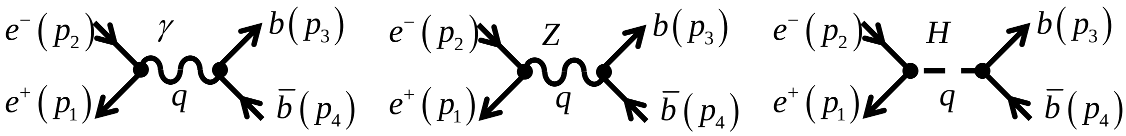
(e) Your propagator may contain terms that look something like $q_\mu q_\nu/M_X^2$. Argue that if we treat the positron and neutrino as massless, this term actually doesn't contribute.

The relevant term contains a factor of $\left[\bar{v}_1\gamma^\mu(1-\gamma_5)u_2\right]q_\mu$, which we can rewrite as

$$\bar{v}_1\not{q}(1-\gamma_5)u_2 = \bar{v}_1(\not{p}_1+\not{p}_2)(1-\gamma_5)u_2 = \bar{v}_1\not{p}_2(1-\gamma_5)u_2 = \bar{v}_1(1+\gamma_5)\not{p}_2 u_2 = 0.$$

26. [30] Bottom quarks can be produced via the process $e^+(p_1)e^-(p_2)\to b(p_3)\bar{b}(p_4)$. Assume we are working at sufficiently high energies that we need to consider all processes, not just QED.

(a) Show that there are three tree diagrams with different intermediate particles that contribute to this process in the standard model.



The three diagrams are sketched above. The possible intermediate particles are the photon, the Z, and the Higgs boson.

(b) Argue that one of them is essentially irrelevant.

The last of these three involves Higgs couplings, which are proportional to the mass of the particles. The electron is very light, so we can ignore this diagram, since the Higgs coupling will be so small.

(c) Write the Feynman amplitude for the other two

Using the rules on the sheet, we have

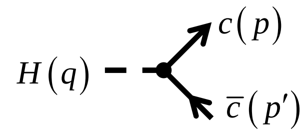
$$i\mathcal{M}=(ie)\left(\tfrac{3}{3}ie\right)\left(\bar{v}_1\gamma^\mu u_2\right)\left(\bar{u}_3\gamma^\nu v_4\right)\frac{-ig_{\mu\nu}}{q^2}+$$

$$\left(\frac{ie}{4\sin\theta_W\cos\theta_W}\right)^2\left[\bar{v}_1\gamma^\mu\left(1-4\sin^2\theta_W-\gamma_5\right)u_2\right]\left[\bar{u}_3\gamma^\nu\left(1-\tfrac{4}{3}\sin^2\theta_W-\gamma_5\right)v_4\right]\frac{i\left(-g_{\mu\nu}+q_\mu q_\nu/M_Z^2\right)}{q^2-M_Z^2}.$$

where the momentum $q=p_1+p_2$. It isn't that hard to show that the $q_\mu q_\nu$ terms vanish because the electrons are nearly massless, but otherwise there is little we can do to simplify it.

$$i\mathcal{M}=ie^2\left\{\frac{\left(\bar{v}_1\gamma^\mu u_2\right)\left(\bar{u}_3\gamma_\mu v_4\right)}{3s}+\frac{\left[\bar{v}_1\gamma^\mu\left(1-4\sin^2\theta_W-\gamma_5\right)u_2\right]\left[\bar{u}_3\gamma_\mu\left(1-\tfrac{8}{3}\sin^2\theta_W-\gamma_5\right)v_4\right]}{16\sin^2\theta_W\cos^2\theta_W\left(s-M_Z^2\right)}\right\}.$$

27. [35] Calculate the decay rate $H \to c\bar{c}$. For definiteness, use $m_H = 126\text{ GeV}$, $m_c = 1.29\text{ GeV}$, and $v = 246\text{ GeV}$. Get both a formula and a number (in MeV) for the process. You may treat the charm quark as massless compared to the Higgs (but don't get zero for the final answer). Don't forget about colors.



The relevant diagram is sketched at right. The amplitude is

$$i\mathcal{M} = -i\frac{m_c}{v}\bar{u}u'$$

We square this and sum over spins to yield

$$\sum|i\mathcal{M}|^2 = \frac{m_c^2}{v^2}\sum(\bar{u}v')(\bar{v}'u) = \frac{m_c^2}{v^2}\sum \mathrm{Tr}(u\bar{u}v'\bar{v}') = \frac{m_c^2}{v^2}\mathrm{Tr}(\not{p}\not{p}') = \frac{m_c^2}{v^2}4p\cdot p'.$$

We then calculate the necessary dot product using the fact that

$$m_H^2 = q^2 = (p+p')^2 = 2p\cdot p'.$$

Substituting this in, we have

$$\sum|i\mathcal{M}|^2 = \frac{2m_H^2 m_c^2}{v^2}.$$

Note that we are summing over final spins; there is no averaging to be done here. We therefore proceed to the final decay rate in the usual way, namely

$$\Gamma = \frac{D}{2M} = \frac{1}{2m_H}\frac{p}{16\pi^2 m_H}\int\sum|i\mathcal{M}|^2 d\Omega = \frac{4\pi p}{32\pi^2 m_H^2}\cdot\frac{2m_c^2 m_H^2}{v^2} = \frac{m_c^2 p}{4\pi v^2}.$$

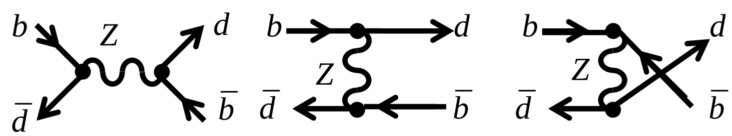
Because the two final particles are massless, their equal momenta will match their energy, and their total energies will be m_H, so $p = \frac{1}{2}m_H$ .Hence the final decay rate is

$$\Gamma = \frac{m_c^2 m_H}{8\pi v^2}.$$

This is to any *one* color; we want the rate to all colors, which throws in a factor of 3. We therefore have (including all colors),

$$\Gamma = \frac{3m_c^2 m_H}{8\pi v^2} = \frac{3(1.29\text{ GeV})^2(126\text{ GeV})}{8\pi(246\text{ GeV})^2} = 4.13\times 10^{-4}\text{ GeV} = 0.413\text{ MeV}.$$

28. [20] The B^0 meson is a bottom – anti-down combination, $b\bar{d}$. It is known experimentally that it can spontaneously change into its anti-particle, $\bar{b}d$.

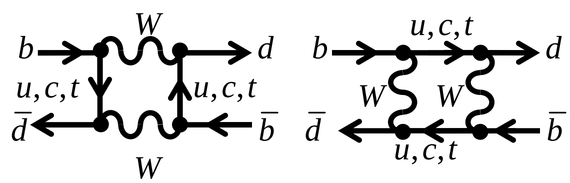


(a) Explain why the tree-level diagrams sketched at right do not contribute.

The *Z*-coupling is diagonal; it cannot connect quarks of different types, which rules out the first two diagrams. The third diagram doesn't even make sense, since it has two fermions both with an arrow into a vertex, and one with two arrows out of the vertex.

(b) Draw at least one one-loop diagram that does contribute to this process

There are two diagrams, as sketched at right. Technically, each of them represents nine diagrams, since there are three possible intermediate up-type quarks for each intermediate line.



29. [25] Suppose that the standard model is right, except that (i) the charges are actually slightly off, and (ii) there are right-handed neutrinos, so that the left-handed fermion content of the theory is given by

$$3\left[\left(3,2,\tfrac{1}{6}+\delta\right)\oplus\left(\overline{3},1,-\tfrac{2}{3}-\delta\right)\oplus\left(\overline{3},1,\tfrac{1}{3}-\delta\right)\oplus\left(1,2,-\tfrac{1}{2}-3\delta\right)\oplus\left(1,1,1+3\delta\right)\oplus\left(1,1,3\delta\right)\right]$$

(a) Find the charge of the left-handed up and down quarks using the $\left(3,2,\tfrac{1}{6}+\delta\right)$, and the charge of the neutrino and electron using $\left(1,2,-\tfrac{1}{2}-3\delta\right)$.

The values of T_3 are $\pm\tfrac{1}{2}$, so the charges are $Q=Y+T_3=\tfrac{1}{6}+\delta\pm\tfrac{1}{2}$ which works out to charges of

$$Q_u=\tfrac{2}{3}+\delta\,,\quad Q_d=-\tfrac{1}{3}+\delta\,.$$

Similarly, for the electron and neutrino, we find $Q=Y+T_3=-\tfrac{1}{2}-3\delta\pm\tfrac{1}{2}$, so

$$Q_\nu=-3\delta\,,\quad Q_e=-1-3\delta\,.$$

(b) Find the charge of the proton, neutron and hydrogen atom.

A proton contains two up and a down quark, a neutron contains two downs and an up quark, and a hydrogen atom is a proton plus an electron.

$$\begin{aligned}Q_p&=2Q_u+Q_d=2\left(\tfrac{2}{3}+\delta\right)+\left(-\tfrac{1}{3}+\delta\right)=1+3\delta\,,\\Q_n&=Q_u+2Q_d=\left(\tfrac{2}{3}+\delta\right)+2\left(-\tfrac{1}{3}+\delta\right)=3\delta\,,\\Q_H&=Q_p+Q_e=0\,.\end{aligned}$$

It is interesting that hydrogen comes out neutral, though this will not be true of any other isotope, since other isotopes will contain neutrons.

(c) Check if the anomalies $\sum Y$ and $\sum YT_3^2$ vanish.

We simply calculate this, keeping track of the number of particles. For T_3^2, we note that anything in a doublet has $T_3^2=\tfrac{1}{4}$, whereas for SU(2) singlets, we have $T_3^2=0$. So we have

$$\begin{aligned}\sum Y&=3\left[6\left(\tfrac{1}{6}+\delta\right)+3\left(-\tfrac{2}{3}-\delta\right)+3\left(\tfrac{1}{3}-\delta\right)+2\left(-\tfrac{1}{2}-3\delta\right)+\left(1+3\delta\right)+\left(3\delta\right)\right]\\&=3\left[1+6\delta-2-3\delta+1-3\delta-1-6\delta+1+3\delta+3\delta\right]=3\cdot 0=0\,,\\\sum YT_3^2&=3\left[6\left(\tfrac{1}{4}\right)\left(\tfrac{1}{6}+\delta\right)+0+0+2\left(\tfrac{1}{4}\right)\left(-\tfrac{1}{2}-3\delta\right)+0+0\right]=3\left[\tfrac{1}{4}+\tfrac{3}{2}\delta-\tfrac{1}{4}-\tfrac{3}{2}\delta\right]=0.\end{aligned}$$

A straightforward but tedious computation will demonstrate that $\sum Y^3=0$ as well.