Solutions to Problems 1

2. It is possible that the universe has small extra dimensions. If so, we should be able to detect them if we use particles with wavelength shorter than the scale L of the extra dimension. Having performed experiments with 4 TeV protons without seeing hints of extra dimensions, estimate the maximum size this extra dimension might be in meters.

We simply use the relationship $\lambda p = 2\pi\hbar = 2\pi$. First we need to find the momentum, which for relativistic neutrinos is effectively the same as the energy. Hence

$$\lambda = \frac{2\pi}{p} = \frac{2\pi\left(0.197 \text{ GeV}\cdot\text{fm}\right)}{4000.0 \text{ GeV}} = 3.09\times10^{-4} \text{ fm} = 3.09\times10^{-19} \text{ m}.$$

I have no idea exactly what the real limit is, but obviously it is pretty short.

5. By July 4, 2012, approximately 5 fb^{-1} of integrated lumonisity at $\sqrt{s} = 8$ TeV had been analyzed by the CMS and ATLAS detectors. How many Higgs particles were produced? The main signal seen was from the process $H \to \gamma\gamma$. The branching ratio for this decay is about 0.25%. How many $H \to \gamma\gamma$ events should have been seen by each experiment?

We need the cross-section, which we get from the link in the previous problem. Looking at **http://arxiv.org/abs/1012.0530**, we see from table 3 that the cross section for Higgs production at $\sqrt{s} = 8$ TeV for a mass of 125 GeV is about 19.81 pb and for a mass of 130 GeV is about 18.34 pb. The numbers look vaguely linear in this region, so using a linear fit, we estimate at 126 GeV the cross-section is about 19.52 pb. We multiply this by the integrated luminosity to obtain

$$N_H = \sigma_H \int L dt = \left(19.52 \text{ pb}\right)\left(5.00 \text{ fb}^{-1}\right) = 97.6\frac{10^{-12} \text{ b}}{10^{-15} \text{ b}} \approx 97,600.$$

According to table 6, the branching ratio to photons is not 0.25%, but 0.23%, so the number of decays is

$$N\left(H \to \gamma\gamma\right) = N_H BR\left(H \to \gamma\gamma\right) = \left(97,600\right)\left(0.0023\right) = 224.$$

This is the approximate number of events expected in each of the two detectors.

7. **Look up the total lifetime of the π^+ and K^+ mesons (summary tables, mesons). What would the rate Γ in GeV be? Then look up the branching ratios in each case to decay to $\mu^+\nu_\mu$. Find the partial rates $\Gamma\left(\pi^+ \to \mu^+\nu_\mu\right)$ and $\Gamma\left(K^+ \to \mu^+\nu_\mu\right)$ in each case, and their ratio.**

According to the particle data book, $\tau_\pi = 2.60\times10^{-8}$ s and $\tau_K = 1.23\times10^{-8}$ s. As noted in the text, $\Gamma = \tau^{-1}$, so we have

$$\Gamma_\pi = \tau_\pi^{-1} = \frac{\hbar}{\tau_\pi} = \frac{6.582\times10^{-16}\ \text{eV}\cdot\text{s}}{2.60\times10^{-8}\ \text{s}} = 2.53\times10^{-8}\ \text{eV} = 2.53\times10^{-17}\ \text{GeV}\,,$$

$$\Gamma_K = \tau_K^{-1} = \frac{\hbar}{\tau_K} = \frac{6.582\times10^{-16}\ \text{eV}\cdot\text{s}}{1.238\times10^{-8}\ \text{s}} = 5.32\times10^{-8}\ \text{eV} = 5.32\times10^{-17}\ \text{GeV}\,.$$

The branching ratio to $\mu^+\nu_\mu$ is 99.99% for the pion and 63.55% for the kaon. We multiply by these numbers to get the partial decay rates:

$$\Gamma\left(\pi^+ \to \mu^+\nu_\mu\right) = \Gamma_\pi BR\left(\pi^+ \to \mu^+\nu_\mu\right) = \left(2.53\times10^{-17}\ \text{GeV}\right)\left(0.9999\right) = 2.53\times10^{-17}\ \text{GeV}\,,$$

$$\Gamma\left(K^+ \to \mu^+\nu_\mu\right) = \Gamma_K BR\left(K^+ \to \mu^+\nu_\mu\right) = \left(5.32\times10^{-17}\ \text{GeV}\right)\left(0.6355\right) = 3.38\times10^{-17}\ \text{GeV}\,,$$

$$\frac{\Gamma\left(\pi^+ \to \mu^+\nu_\mu\right)}{\Gamma\left(K^+ \to \mu^+\nu_\mu\right)} = \frac{2.53}{3.38} = 0.749\,.$$

As we will later realize, the denominator is larger overwhelmingly because of the higher mass of the kaon.

8. **Find a dimensionless combination of *e*, ε_0, $\hbar$ and *c*. Then, setting $\varepsilon_0 = \hbar = c = 1$, find the dimensionless value of the fundamental charge *e*.**

The values of these constants are

$$e = 1.602\times10^{-19}\ \text{C},$$
$$\varepsilon_0 = 8.854\times10^{-12}\ \text{C}^2\text{s}^2/\text{m}^3/\text{kg}\,,$$
$$\hbar = 1.055\times10^{-34}\ \text{kg}\cdot\text{m}^2/\text{s}\,,$$
$$c = 2.998\times10^{8}\ \text{m/s}\,.$$

We note that *e* has C in it, and ε_0 has C^2, so if we square the former and divide by the latter, the C will cancel out. This will leave kg in the numerator, but if we divide by $\hbar$, that will go away, and it isn't hard to see that if you then divide by *c*, you get something dimensionless. So we have

$$\frac{e^2}{\varepsilon_0 \hbar c} = \frac{\left(1.602\times10^{-19}\text{ C}\right)^2}{\left(8.854\times10^{-12}\text{ C}^2\text{s}^2/\text{m}^3/\text{kg}\right)\left(1.055\times10^{-34}\text{ kg}\cdot\text{m}^2/\text{s}\right)\left(2.998\times10^{8}\text{ m/s}\right)} = 0.09164\,.$$

The quantity on the right is true in any units, but in particle physics units, the factors in the denominator on the left are 1, and this formula simplifies to $e^2 = 0.09245$, or taking the square root,

$$e = \sqrt{0.09164} = 0.3027\,.$$

We could have gotten the same answer starting from the equation on the inside front cover $e^2/4\pi = 1/137.04$

9. By using a suitable combination of $\hbar$ and *c*, write Newton's constant in the form $G_N = M_P^n$, where M_P has units of mass or energy, and is called the *Planck mass*. Determine the integer *n* and the value of M_P in GeV. If a proton collider were operating at $E = M_P$ and used $B = 10\text{ T}$ magnets, what would be its radius in light-years?

The relevant constants are

$$G_N = 6.674\times10^{-11}\text{ m}^3/\text{kg/s}^2\,,$$
$$\hbar = 1.055\times10^{-34}\text{ kg}\cdot\text{m}^2/\text{s}\,,$$
$$c = 2.998\times10^{8}\text{ m/s}\,.$$

Let's start by writing it as a mass, which we can do if we get rid of all the meters and seconds. It is clear that $G_N/\hbar$ will have only one factor of m in the numerator and one factor of s in the denominator, so if we then divide by *c* we get

$$\frac{G_N}{\hbar c} = \frac{6.674\times10^{-11}\text{ m}^3/\text{kg/s}^2}{\left(1.055\times10^{-34}\text{ kg}\cdot\text{m}^2/\text{s}\right)\left(2.998\times10^{8}\text{ m/s}\right)} = 2.11\times10^{15}\text{ kg}^{-2}\,.$$

Since it is mass to the minus two, we write it in the form M_P^{-2}, so that by definition,

$$M_P^{-2} \equiv \frac{G_N}{\hbar c} = 2.11\times10^{15}\text{ kg}^{-2}\,,$$
$$M_P = \left(2.11\times10^{15}\text{ kg}^{-2}\right)^{-1/2} = 2.177\times10^{-8}\text{ kg}\,.$$

In particle physics units, we would write this as $G = M_P^{-2}$. We then convert this to an energy by using the conversion $1\text{ kg} = 5.6\times10^{26}\text{ GeV}$, so

$$M_P = \left(2.177\times10^{-8}\text{ kg}\right)\left(5.6\times10^{26}\text{ GeV/kg}\right) = 1.22\times10^{19}\text{ GeV}\,.$$

We can then find how large a collider we need to reach this energy using eq. (1.5). The momentum is effectively the same as the energy, and we are working in units where $c = 1$, so

$$p = \left(\frac{B}{\mathrm{T}}\right)\left(\frac{R}{\mathrm{km}}\right)\left(299.8\ \mathrm{GeV}\right),$$

$$R = \frac{1.22\times 10^{19}}{299.8\cdot 10}\,\mathrm{km} = 4.066\times 10^{15}\ \mathrm{km} = \frac{\left(4.066\times 10^{15}\ \mathrm{km}\right)c}{\left(2.998\times 10^{5}\ \mathrm{km/s}\right)\left(3.156\times 10^{7}\,\mathrm{s/y}\right)} \approx 430\ \mathrm{ly}\,.$$

It is inconceivable we will ever reach this energy without new technology.

14. Perform the following integrals:

(a) $\int_0^\infty E^n \delta\left(E^2 - \mathbf{p}^2 - m^2\right)$ **for arbitrary n.**

The argument of the delta function vanishes at $E = \sqrt{\mathbf{p}^2 + m^2}$, so

$$\int_0^\infty E^n \delta\left(E^2 - \mathbf{p}^2 - m^2\right) dE = \left.\frac{E^n}{2E}\right|_{E=\sqrt{\mathbf{p}^2+m^2}} = \frac{1}{2}\left(\mathbf{p}^2 + m^2\right)^{\frac{1}{2}(n-1)}\,.$$

(b) $\int_0^\infty dE_1 \int_0^\infty dE_2\, \theta\left(\tfrac{1}{2}m - E_1\right)\theta\left(\tfrac{1}{2}m - E_2\right)\theta\left(E_1 + E_2 - \tfrac{1}{2}m\right)\left(\tfrac{1}{2}m^2 E_1 - mE_1^2\right)$

As we argued in class, the first two Heaviside functions restrict the upper limit on the energy integrals to each be $\tfrac{1}{2}m$, and the third one demands that $E_1 + E_2 > \tfrac{1}{2}m$. If we let the inner integral be the E_2 integral, then we have

$$\int_0^\infty dE_1 \int_0^\infty dE_2\, \theta\left(\tfrac{1}{2}m - E_1\right)\theta\left(\tfrac{1}{2}m - E_2\right)\theta\left(E_1 + E_2 - \tfrac{1}{2}m\right)\left(\tfrac{1}{2}m^2 E_1 - mE_1^2\right)$$
$$= \int_0^{\frac{1}{2}m} dE_1 \int_{\frac{1}{2}m - E_1}^{\frac{1}{2}m} dE_2 \left(\tfrac{1}{2}m^2 E_1 - mE_1^2\right) = \int_0^{\frac{1}{2}m} dE_1 \left(\tfrac{1}{2}m^2 E_1 - mE_1^2\right) E_2\Big|_{\frac{1}{2}m - E_1}^{\frac{1}{2}m}$$
$$= \int_0^{\frac{1}{2}m} dE_1 \left(\tfrac{1}{2}m^2 E_1^2 - mE_1^3\right) = \left.\left(\tfrac{1}{6}m^2 E_1^3 - \tfrac{1}{4}mE_1^4\right)\right|_0^{\frac{1}{2}m} = \tfrac{1}{48}m^5 - \tfrac{1}{64}m^5 = \tfrac{1}{192}m^5\,.$$

(c) $\int\left[\left(1-2\sin^2\theta_W\right)^2 E^4+\sin^4\theta_W E^4\left(1+\cos\theta\right)^2\right]d\Omega$ **(note: θ_W is a constant)**

There is an integral $\int_0^{2\pi} d\phi = 2\pi$, which is easy, and an integral over θ, which I like to do by changing to $z=\cos\theta$

$$\begin{aligned}&\int\left[\left(1-2\sin^2\theta_W\right)^2 E^4+\sin^4\theta_W E^4\left(1+\cos\theta\right)^2\right]d\Omega\\
&=2\pi E^4\int_{-1}^{1}dz\left[\left(1-2\sin^2\theta_W\right)^2+\sin^4\theta_W\left(1+2z+z^2\right)\right]\\
&=2\pi E^4\left[\left(1-2\sin^2\theta_W\right)^2 z+\sin^4\theta_W\left(z+z^2+\tfrac{1}{3}z^3\right)\right]_{-1}^{1}\\
&=2\pi E^4\left[2\left(1-2\sin^2\theta_W\right)^2+\sin^4\theta_W\left(2+0+\tfrac{2}{3}\right)\right]=4\pi E^4\left(1-4\sin^2\theta_W+\tfrac{16}{3}\sin^4\theta_W\right).\end{aligned}$$

It is unclear what further simplification is desirable.

(d) $\int\frac{g^4 p\cos^2\theta}{128\pi^2 E\left(E^2-p^2\cos^2\theta\right)}d\Omega$

$$\begin{aligned}&\int\frac{g^4 p\cos^2\theta}{128\pi^2 E\left(E^2-p^2\cos^2\theta\right)}d\Omega=\frac{2\pi g^4}{128\pi^2 Ep}\int_{-1}^{1}\frac{p^2\cos^2\theta\, d\cos\theta}{E^2-p^2\cos^2\theta}\\
&=\frac{g^4}{32\pi Ep}\int_0^1\left[-1+\frac{E^2}{E^2-p^2\cos^2\theta}\right]d\cos\theta=\frac{g^4}{32\pi Ep}\left[-1+\frac{1}{2}\int_0^1\left(\frac{E\, d\cos\theta}{E+p\cos\theta}+\frac{E\, d\cos\theta}{E-p\cos\theta}\right)\right]\\
&=\frac{g^4}{32\pi Ep}\left\{-1+\frac{E}{2p}\left[\ln\left(E+p\cos\theta\right)-\ln\left(E-p\cos\theta\right)\right]_0^1\right\}=\frac{g^4}{32\pi p^2}\left\{\frac{1}{2}\ln\left(\frac{E+p}{E-p}\right)-\frac{p}{E}\right\}.\end{aligned}$$

The logarithm term can be written more succinctly as $\tanh^{-1}\left(p/E\right)$ if we want.