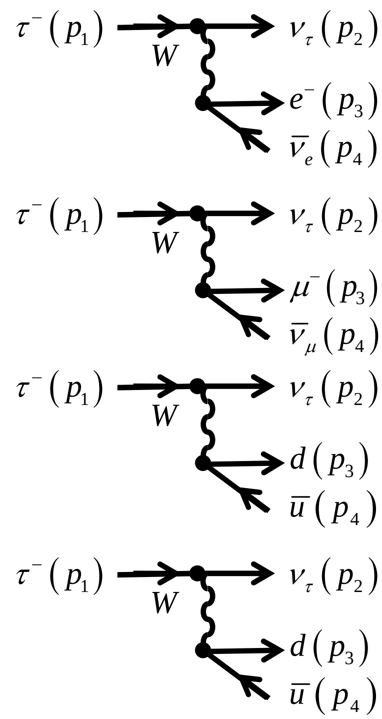
Solutions to Problems 10a

1. Find the following decay rates for the τ^- lepton: $\Gamma\left(\tau^- \to \nu_\tau e^- \bar{\nu}_e\right)$, $\Gamma\left(\tau^- \to \nu_\tau \mu^- \bar{\nu}_\mu\right)$, $\Gamma\left(\tau^- \to \nu_\tau d\bar{u}\right)$, $\Gamma\left(\tau^- \to \nu_\tau s\bar{u}\right)$ (don't forget colors). Assume all final particles are massless. Calculate the total decay rate, and the branching ratio for each of the first two decays, and compare with the experimental values. The decay rates are easily computed with minor modifications of eq. (10.15). Don't be bothered if you don't get exactly the right answer.



The diagrams all look very similar, and are sketched at right. For the electron and the muon in the final states, the amplitudes are identical, namely

$$i\mathcal{M} = \left(\frac{-ig}{2\sqrt{2}}\right)^2 \frac{i}{M_W^2}\left[\bar{u}_2\gamma^\mu\left(1-\gamma_5\right)u_1\right]\left[\bar{u}_3\gamma_\mu\left(1-\gamma_5\right)v_4\right].$$

For the other two diagrams, there will be an additional factor for the CKM matrix, so we will get

$$i\mathcal{M}\left(\tau^- \to \nu_\tau d\bar{u}\right) = \left(\frac{-ig}{2\sqrt{2}}\right)^2 V_{ud}^* \frac{i}{M_W^2}\left[\bar{u}_2\gamma^\mu\left(1-\gamma_5\right)u_1\right]\left[\bar{u}_3\gamma_\mu\left(1-\gamma_5\right)v_4\right],$$

$$i\mathcal{M}\left(\tau^- \to \nu_\tau s\bar{u}\right) = \left(\frac{-ig}{2\sqrt{2}}\right)^2 V_{us}^* \frac{i}{M_W^2}\left[\bar{u}_2\gamma^\mu\left(1-\gamma_5\right)u_1\right]\left[\bar{u}_3\gamma_\mu\left(1-\gamma_5\right)v_4\right].$$

It should be understood that these two amplitudes are correct assuming the quarks have identical color. The calculation will be identical with the notes, except that the muon mass will be replaced by the tau mass. The quark decays, of course, will have the CKM matrix elements squared as well. We therefore have

$$\Gamma\left(\tau^- \to \nu_\tau e^- \bar{\nu}_e\right) = \Gamma\left(\tau^- \to \nu_\tau \mu^- \bar{\nu}_\mu\right) = \frac{1}{192\pi^3} G_F^2 m_\tau^5\ ,$$

$$\Gamma\left(\tau^- \to \nu_\tau d\bar{u}\right) = \frac{1}{192\pi^3} G_F^2 m_\tau^5 \left|V_{ud}\right|^2\ , \quad \Gamma\left(\tau^- \to \nu_\tau d\bar{s}\right) = \frac{1}{192\pi^3} G_F^2 m_\tau^5 \left|V_{us}\right|^2\ .$$

The last two equations are if you were going to a particular color. So we need to triple to account for the three colors, so

$$\Gamma\left(\tau^- \to \nu_\tau d\bar{u}\right) = \frac{1}{64\pi^3} G_F^2 m_\tau^5 \left|V_{ud}\right|^2\ , \quad \Gamma\left(\tau^- \to \nu_\tau d\bar{s}\right) = \frac{1}{64\pi^3} G_F^2 m_\tau^5 \left|V_{us}\right|^2\ .$$

The total decay rate, therefore, will be

$$\Gamma_{\text{tot}} = \frac{1}{192\pi^3} G_F^2 m_\tau^5 \left(1+1+3\left|V_{ud}\right|^2 + 3\left|V_{us}\right|^2\right).$$

We can estimate the relevant matrix elements using equation (10.71) together with (10.62) to yield

$$V_{ud}V_{ud}^* + V_{us}V_{us}^* = 1 - V_{ub}V_{ub}^* = 1 - \left|V_{ub}\right|^2 = 1 - 0.00351^2 = 0.99999\,.$$

To far more accuracy than we need, we can just approximate this as one. We therefore have

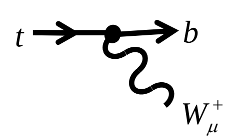
$$\Gamma_{\text{tot}} \approx \frac{5}{192\pi^3} G_F^2 m_\tau^5 = \frac{5}{192\pi^3}\left(1.16637\times10^{-5}\ \text{GeV}^{-2}\right)^2\left(1.77682\ \text{GeV}\right)^5 = 2.0235\times10^{-12}\ \text{GeV}$$
$$= \frac{2.0235\times10^{-3}\ \text{eV}}{6.582\times10^{-16}\ \text{eV}\cdot\text{s}} = 3.074\times10^{12}\ \text{s}^{-1}\,.$$

The lifetime will therefore be $\tau = \Gamma_{\text{tot}}^{-1} = 3.25\times10^{-13}\ \text{s}$. The actual lifetime is about $\tau = 2.90\times10^{-13}\ \text{s}$, about a 12% error, probably mostly because we are pretending the final hadronic states are all massless. The branching ratio will be predicted to be

$$BR\left(\tau^- \to \nu_\tau e^- \bar{\nu}_e\right) = \frac{\Gamma\left(\tau^- \to \nu_\tau e^- \bar{\nu}_e\right)}{\Gamma_{\text{tot}}} = \frac{1}{5}\,, \quad BR\left(\mu^- \to \nu_\tau \mu^- \bar{\nu}_\mu\right) = \frac{\Gamma\left(\tau^- \to \nu_\tau \mu^- \bar{\nu}_\mu\right)}{\Gamma_{\text{tot}}} = \frac{1}{5}\,.$$

Though it isn't correct, it isn't terribly far from the experimental value of 17.83% and 17.41% respectively. In fact, these are off by essentially the same factor, because the leptonic decays are accurately predicted (though the muon decay should be corrected for the finite muon mass).

3. Find the decay rate for top decay, $t \to W^+ b$, neglecting the bottom mass, but including the *W*-mass, in GeV. Note that even though this is considered a "weak" decay, it has a very large rate (a GeV rate is faster than typical strong interactions).



There is only one process, sketched at right. The Feynman amplitude will be

$$i\mathcal{M} = \frac{-ig}{2\sqrt{2}} V_{tb}^* \left[\bar{u}_b \gamma^\mu \left(1-\gamma_5\right) u_t\right] \varepsilon_\mu^*$$

We multiply this by its complex conjugate in the usual way to yield

$$\left|i\mathcal{M}\right|^2 = \tfrac{1}{8} g^2 \left|V_{tb}\right|^2 \varepsilon_\mu^* \varepsilon_\nu \left[\bar{u}_b \gamma^\mu \left(1-\gamma_5\right) u_t\right]\left[\bar{u}_t \left(1+\gamma_5\right)\gamma^\nu u_b\right]$$

We then sum over final spin and final polarization and average over the initial spin, keeping in mind that we are treating the bottom quark as massless, and we obtain

$$\tfrac{1}{2}\sum\left|i\mathcal{M}\right|^2 = \tfrac{1}{16} g^2 \left|V_{tb}\right|^2 \left(-g_{\mu\nu} + q_\mu q_\nu / M_W^2\right) \text{Tr}\left[\not{p}_b \gamma^\mu \left(1-\gamma_5\right)\left(\not{p}_t + m_t\right)\left(1+\gamma_5\right)\gamma^\nu\right]$$

It isn't hard to see that $\left(1-\gamma_5\right)\left(1+\gamma_5\right) = 0$ and $\left(1-\gamma_5\right)\not{p}_t\left(1+\gamma_5\right) = \left(1-\gamma_5\right)^2 \not{p}_t = 2\left(1-\gamma_5\right)\not{p}_t$, so we have

$$\begin{aligned}
\tfrac{1}{2}\sum\left|i\mathcal{M}\right|^2 &= \tfrac{1}{8}g^2\left|V_{tb}\right|^2\left(-g_{\mu\nu}+q_\mu q_\nu\big/M_W^2\right)\mathrm{Tr}\left[\not{p}_b\gamma^\mu\left(1-\gamma_5\right)\not{p}_t\gamma^\nu\right] \\
&= \tfrac{1}{8}g^2\left|V_{tb}\right|^2\,\mathrm{Tr}\left[\left(1-\gamma_5\right)\left(-\not{p}_b\gamma^\mu\not{p}_t\gamma_\mu+M_W^{-2}\not{p}_b\not{q}\not{p}_t\not{q}\right)\right] \\
&= \tfrac{1}{8}g^2\left|V_{tb}\right|^2\,\mathrm{Tr}\left[\left(1-\gamma_5\right)\left(2\not{p}_b\not{p}_t+M_W^{-2}\not{p}_b\not{q}\not{p}_t\not{q}\right)\right] \\
&= \tfrac{1}{2}g^2\left|V_{tb}\right|^2\left\{2p_b\cdot p_t+M_W^{-2}\left[2\left(p_b\cdot q\right)\left(p_t\cdot q\right)-\left(p_b\cdot p_t\right)q^2+i\varepsilon^{\alpha\beta\mu\nu}p_{b\alpha}q_\beta p_{t\mu}q_\nu\right]\right\} \\
&= g^2\left|V_{tb}\right|^2\left\{\tfrac{1}{2}p_b\cdot p_t+\left(p_b\cdot q\right)\left(p_t\cdot q\right)\big/M_W^2\right\}.
\end{aligned}$$

We now need to work out all the dot products. The momenta satisfy $p_t = p_b + q$. We can square this directly, or rearrange it in a variety of ways to demonstrate that

$$\begin{aligned}
m_t^2 = p_t^2 = \left(p_b+q\right)^2 = 2p_b\cdot q+M_W^2\,, &\quad \text{so} \quad p_b\cdot q=\tfrac{1}{2}\left(m_t^2-M_W^2\right), \\
0 = p_b^2 = \left(p_t-q\right)^2 = m_t^2+M_W^2-2p_t\cdot q, &\quad \text{so} \quad p_t\cdot q=\tfrac{1}{2}\left(m_t^2+M_W^2\right), \\
M_W^2 = q^2 = \left(p_t-p_b\right)^2 = m_t^2-2p_t\cdot p_b\,, &\quad \text{so} \quad p_t\cdot p_b=\tfrac{1}{2}\left(m_t^2-M_W^2\right).
\end{aligned}$$

Substituting these expressions in, we have

$$\tfrac{1}{2}\sum\left|i\mathcal{M}\right|^2 = \frac{g^2}{4}\left|V_{tb}\right|^2\left[m_t^2-M_W^2+\frac{\left(m_t^2-M_W^2\right)\left(m_t^2+M_W^2\right)}{M_W^2}\right] = \sqrt{2}G_F\left|V_{tb}\right|^2\left(m_t^2-M_W^2\right)\left(m_t^2+2M_W^2\right)$$

where we used $G_F = g^2\big/4\sqrt{2}M_W^2$ at the last step. We now go through the usual steps to get the decay rate, namely

$$\Gamma = \frac{D}{2m_t} = \frac{1}{2m_t}\cdot\frac{p}{16\pi^2 m_t}\int\tfrac{1}{2}\sum\left|i\mathcal{M}\right|^2 d\Omega = \frac{4\pi p}{32\pi^2 m_t^2}\sqrt{2}G_F\left|V_{tb}\right|^2\left(m_t^2-M_W^2\right)\left(m_t^2+2M_W^2\right).$$

The momentum is the same as the energy of the *b*-quark, which can be found from $p_t\cdot p_b = m_t E_b = \tfrac{1}{2}\left(m_t^2-M_W^2\right)$. Substituting this in, we have

$$\Gamma = \frac{1}{8\sqrt{2}\pi m_t^3}G_F\left|V_{tb}\right|^2\left(m_t^2-M_W^2\right)^2\left(m_t^2+2M_W^2\right).$$

This process is sometimes referred to as semi-weak, since it has only one factor of G_F. Numerically, its value works out to

$$\begin{aligned}
\Gamma &= \frac{1.166\times10^{-5}\ \mathrm{GeV}^{-2}}{8\sqrt{2}\pi\left(172.9\ \mathrm{GeV}\right)^3}\left(0.999\right)^2\left[\left(172.9\ \mathrm{GeV}\right)^2-\left(80.4\ \mathrm{GeV}\right)^2\right]^2 \\
&\times\left[\left(172.9\ \mathrm{GeV}\right)^2+2\left(80.4\ \mathrm{GeV}\right)^2\right] \\
&= 1.49\ \mathrm{GeV}.
\end{aligned}$$

This very large decay rate makes it clear that “weak” interactions are not always weak.