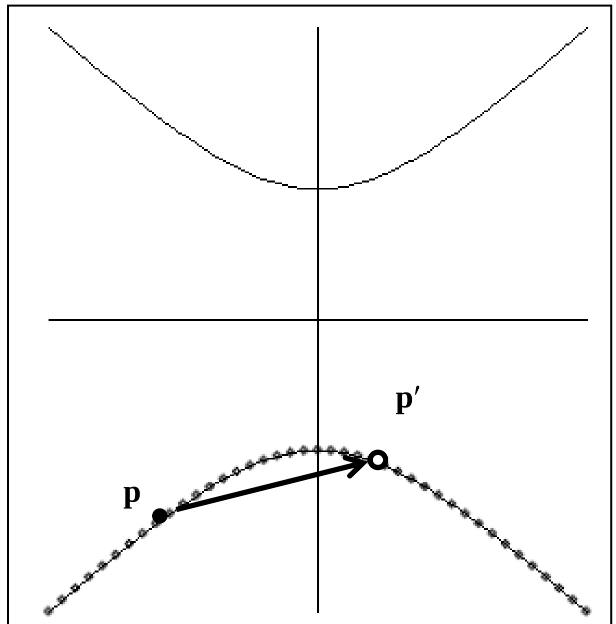
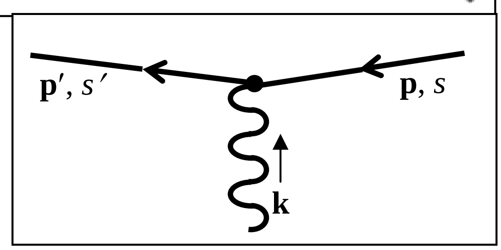
Solutions to Problems 3b

9. [10] In the lecture notes, we discussed what "scattering" really means in a variety of situations, such as positive energy → positive energy (scattering), negative → positive (pair creation), and positive → negative (pair annihilation). The only case we haven't discussed is what it means when negative energy → negative energy. Let $-p$ be the momentum of an electron that scatters into a state of momentum $-p'$. How do we interpret this in the hole picture? What do we see in the initial state, and in the final state? Draw both kinds of pictures (filled/empty state diagrams and Feynman-like diagrams) to show what is going on. What is the probability after time *T* that such a transition will have occurred, to linear order in the field?



In this process, an electron of four-momentum $-p$ becomes an electron of momentum $-p'$. Since these are both negative energy states, they should normally be both filled. Since we scattered to a state of momentum $-p'$, this state must not have been filled to begin with. Since we scattered from a state of momentum $-p$, this state will end up empty in the end. Therefore the initial state is actually a positron of momentum p', and the final state is a positron of momentum *p*. The filled/empty state diagram looks like the sketch at right. The Feynman-like diagram is shown below it. We can derive the probability by modifying eq. (3.65) by replacing each of the momenta by their negatives, and by replacing each of the *u*'s with *v*'s, to yield



$$P(I \to F) = (2\pi)^4 \frac{T}{4EE'V} \delta^4(p - p' - k) \left| \bar{v}(\mathbf{p}', s') \not{a} v(\mathbf{p}, s) \right|^2 .$$

10. [15] Alice and Bob are doing electromagnetic scattering from an electromagnetic field given by eq. (3.63); however, they are working in different gauges, so $A_\mu^A(x) = a_\mu^A e^{-ik\cdot x}$ and $A_\mu^B(x) = a_\mu^B e^{-ik\cdot x}$.

(a) [3] Given that they are related by a gauge transformation, eq. (3.56a), what can we say about the relation between a_μ^A and a_μ^B?

As discussed in class, they must be related by a Gauge transformation eq. (3.56) with a gauge function proportional to $e^{-ik\cdot x}$. We therefore have

$$A_\mu^B(x) = A_\mu^A(x) + \partial_\mu \chi(x) \quad \text{with} \quad \chi(x) = ce^{-ik\cdot x} ,$$
$$a_\mu^b e^{-ik\cdot x} = a_\mu^a e^{-ik\cdot x} + c\partial_\mu e^{-ik\cdot x} = a_\mu^a e^{-ik\cdot x} - ick_\mu e^{-ik\cdot x} ,$$
$$a_\mu^b = a_\mu^a - ick_\mu .$$

The factor of i is essentially irrelevant, since c is an arbitrary complex constant.

(b) [6] Show that if they both use eq. (3.65), they will nonetheless get the same answer.

The amplitudes inside the absolute values in eq. (3.65) will be related by

$$\bar{u}(\mathbf{p}',s')\not{a}^B u(\mathbf{p},s) - \bar{u}(\mathbf{p}',s')\not{a}^A u(\mathbf{p},s) = \bar{u}(\mathbf{p}',s')\gamma^\mu\left(a_\mu^A - a_\mu^B\right)u(\mathbf{p},s) = -ic\bar{u}(\mathbf{p}',s')\not{k}u(\mathbf{p},s)$$

Now, since this is multiplied by a Dirac delta function, it gets multiplied by zero (which automatically makes it true) unless $p' = p + k$, so we replace $k = p' - p$ and then use eqs. (3.48) to yield

$$\bar{u}(\mathbf{p}',s')\not{k}u(\mathbf{p},s) = \bar{u}(\mathbf{p}',s')\left(\not{p}' - \not{p}\right)u(\mathbf{p},s) = \bar{u}(\mathbf{p}',s')\not{p}'u(\mathbf{p},s) - \bar{u}(\mathbf{p}',s')\not{p}u(\mathbf{p},s)$$
$$= \bar{u}(\mathbf{p}',s')mu(\mathbf{p},s) - \bar{u}(\mathbf{p}',s')mu(\mathbf{p},s) = 0.$$

They therefore will get the same result.

(c) [6] Show that they also get the same result if they are doing pair production (eq. (3.66)) or pair annihilation (eq. (3.67)).

In each case, we relate the amplitudes and then use the Dirac delta function to replace k with some combination of p and p', and finally use some combination of eqs. (3.48) and (3.49) to simplify. For eq. (3.66), we have

$$\bar{u}(\mathbf{p}',s')\not{a}^B v(\mathbf{p},s) - \bar{u}(\mathbf{p}',s')\not{a}^A v(\mathbf{p},s) = -ic\bar{u}(\mathbf{p}',s')\not{k}v(\mathbf{p},s) ,$$
$$\bar{u}(\mathbf{p}',s')\not{k}v(\mathbf{p},s) = \bar{u}(\mathbf{p}',s')\left(\not{p}' + \not{p}\right)v(\mathbf{p},s) = \bar{u}(\mathbf{p}',s')mv(\mathbf{p},s) - \bar{u}(\mathbf{p}',s')mv(\mathbf{p},s) = 0 .$$

For eq. (3.67), we have

$$\bar{v}(\mathbf{p}',s')\not{a}^B u(\mathbf{p},s) - \bar{v}(\mathbf{p}',s')\not{a}^A u(\mathbf{p},s) = ic\bar{v}(\mathbf{p}',s')\not{k}u(\mathbf{p},s)$$
$$\bar{v}(\mathbf{p}',s')\not{k}u(\mathbf{p},s) = -\bar{v}(\mathbf{p}',s')\left(\not{p}' + \not{p}\right)u(\mathbf{p},s) = \bar{v}(\mathbf{p}',s')mu(\mathbf{p},s) - \bar{v}(\mathbf{p}',s')mu(\mathbf{p},s) = 0 .$$