Solutions to Problems 4b

6. Consider a renormalizable theory with two charged spin 0 particles, ψ_1 with charge Q_1 and ψ_2 with charge Q_2. They are not equivalent to their anti-particles ψ_1^* and ψ_2^*.

(a) Write down all possible renormalizable matrix elements of the form $\langle 0|\mathcal{H}|X\rangle$, where *X* has more than two particles, if $Q_2 = Q_1$, and figure out which ones must be real.

We will write all particles over on the right. We simply have to have as many particles as anti-particles, since the charges of each is the same. We are interested only in those with at least three particles and at most four, but there is no way you can make the charge balance with just three. The renormalizable matrix elements will be

$$\langle 0|\mathcal{H}|\psi_1\psi_1\psi_1^*\psi_1^*\rangle, \quad \langle 0|\mathcal{H}|\psi_1\psi_1\psi_1^*\psi_2^*\rangle, \quad \langle 0|\mathcal{H}|\psi_1\psi_1\psi_2^*\psi_2^*\rangle,$$
$$\langle 0|\mathcal{H}|\psi_1\psi_2\psi_1^*\psi_1^*\rangle, \quad \langle 0|\mathcal{H}|\psi_1\psi_2\psi_1^*\psi_2^*\rangle, \quad \langle 0|\mathcal{H}|\psi_1\psi_2\psi_2^*\psi_2^*\rangle,$$
$$\langle 0|\mathcal{H}|\psi_2\psi_2\psi_1^*\psi_1^*\rangle, \quad \langle 0|\mathcal{H}|\psi_2\psi_2\psi_1^*\psi_2^*\rangle, \quad \langle 0|\mathcal{H}|\psi_2\psi_2\psi_2^*\psi_2^*\rangle.$$

To see if they are real, take the Hermitian conjugate (which moves them all to the other side), then bring them back to the right-hand side by changing all particles to anti-particles. If you end up with the same thing, then the quantity is real. We find that the three on the diagonal are real:

$$\langle 0|\mathcal{H}|\psi_1\psi_1\psi_1^*\psi_1^*\rangle, \quad \langle 0|\mathcal{H}|\psi_1\psi_2\psi_1^*\psi_2^*\rangle, \quad \langle 0|\mathcal{H}|\psi_2\psi_2\psi_2^*\psi_2^*\rangle.$$

The other six are complex, but they come in pairs. Roughly speaking, the nine matrix elements above form a rank three Hermitian matrix, so we have

$$\langle 0|\mathcal{H}|\psi_1\psi_1\psi_1^*\psi_2^*\rangle = \langle 0|\mathcal{H}|\psi_1\psi_2\psi_1^*\psi_1^*\rangle^*,$$
$$\langle 0|\mathcal{H}|\psi_1\psi_1\psi_2^*\psi_2^*\rangle = \langle 0|\mathcal{H}|\psi_2\psi_2\psi_1^*\psi_1^*\rangle^*,$$
$$\langle 0|\mathcal{H}|\psi_1\psi_2\psi_2^*\psi_2^*\rangle = \langle 0|\mathcal{H}|\psi_2\psi_2\psi_1^*\psi_2^*\rangle^*.$$

(b) Suppose instead that $Q_2 = -Q_1$. Show that you don't have to redo the work of part (a) (hint: it is arbitrary what we call a particle and what we call an anti-particle).

It is pretty easy to see that if ψ_2 has the negative of the charge of ψ_1, then ψ_2^* has the same charge as ψ_1. Hence we can simply replace ψ_2 with ψ_2^* in our previous answers. For example, there are nine basic matrix elements, which are

$$\langle 0|\mathcal{H}|\psi_1\psi_1\psi_1^*\psi_1^*\rangle,\quad \langle 0|\mathcal{H}|\psi_1\psi_1\psi_1^*\psi_2\rangle,\quad \langle 0|\mathcal{H}|\psi_1\psi_1\psi_2\psi_2\rangle,$$
$$\langle 0|\mathcal{H}|\psi_1\psi_2^*\psi_1^*\psi_1^*\rangle,\quad \langle 0|\mathcal{H}|\psi_1\psi_2^*\psi_1^*\psi_2\rangle,\quad \langle 0|\mathcal{H}|\psi_1\psi_2^*\psi_2\psi_2\rangle,$$
$$\langle 0|\mathcal{H}|\psi_2^*\psi_2^*\psi_1^*\psi_1^*\rangle,\quad \langle 0|\mathcal{H}|\psi_2^*\psi_2^*\psi_1^*\psi_2\rangle,\quad \langle 0|\mathcal{H}|\psi_2^*\psi_2^*\psi_2\psi_2\rangle.$$

The three on the diagonal are real, the others come in complex pairs.

(c) Repeat part (a) if $Q_2 = 2Q_1$.

This time you can have either pairs of ψ_1 and ψ_1^*, or ψ_2 and ψ_2^*, or you can exchange two ψ_1's for a ψ_2. It isn't too hard to figure out that the only non-zero basic matrix elements are

$$\langle 0|\mathcal{H}|\psi_1\psi_1\psi_1^*\psi_1^*\rangle,\quad \langle 0|\mathcal{H}|\psi_1\psi_2\psi_1^*\psi_2^*\rangle,\quad \langle 0|\mathcal{H}|\psi_2\psi_2\psi_2^*\psi_2^*\rangle,\quad \langle 0|\mathcal{H}|\psi_2\psi_1^*\psi_1^*\rangle,\quad \langle 0|\mathcal{H}|\psi_1\psi_1\psi_2^*\rangle.$$

The Hermitian property lets you bring them all to the other side, and then you use the anti-particle property to bring them all back to the right. This turns all particles into anti-particles, and then if you end up with the same thing, the expression is real. We find this way that the first htree expressions are real, and the last two are complex conjugates of each other.

(d) Repeat part (a) if $Q_2 = 5Q_1$. Argue that the number of particles (minus anti-particles) for ψ_1 and ψ_2 are separately conserved. Such a conservation law that is demanded by renormalizability is called an *accidental* symmetry. Baryon number in the Standard Model is an accidental symmetry.

If you have an excess of ψ_2 or ψ_2^*, you would have to balance it out by including five ψ_1's or ψ_1^*'s, which would not be renormalizable. Hence the only combinations allowed are

$$\langle 0|\mathcal{H}|\psi_1\psi_1\psi_1^*\psi_1^*\rangle,\quad \langle 0|\mathcal{H}|\psi_1\psi_2\psi_1^*\psi_2^*\rangle,\quad \langle 0|\mathcal{H}|\psi_2\psi_2\psi_2^*\psi_2^*\rangle\ .$$

These will all be real. In each case, the separate ψ_1 number and ψ_2 number are conserved. This will be true if you move some of the particles to the other side using the anti-particle property as well, so ultimately each of these quantum numbers is separately conserved.

8. Suppose two particles are moving along the x^3 axis, so that $p_1^\mu = (E_1, 0, 0, p_1)$ and $p_2^\mu = (E_2, 0, 0, p_2)$. Note that one particle might be at rest. Show that the combination $|E_2\mathbf{p}_1 - E_1\mathbf{p}_2|$ is unaffected by a boost along the x^3-axis.

When we perform such a boost, the momenta become

$$p_1'^\mu = \Lambda^\mu{}_\nu p_1^\nu\ ,\quad p_2'^\mu = \Lambda^\mu{}_\nu p_2^\nu\ .$$

Writing this out explicitly, we find

$$E_1' = E_1 \cosh\phi - p_1 \sinh\phi\,, \quad E_2' = E_2 \cosh\phi - p_2 \sinh\phi\,,$$
$$p_1' = p_1 \cosh\phi - E_1 \sinh\phi, \quad p_2' = p_2 \cosh\phi - E_2 \sinh\phi.$$

The only non-zero component is the *z*-component, so we have

$$\begin{aligned}
E_2' p_1 - E_1' p_2' &= \left(E_2 \cosh\phi - p_2 \sinh\phi\right)\left(p_1 \cosh\phi - E_1 \sinh\phi\right) \\
&\quad -\left(E_1 \cosh\phi - p_1 \sinh\phi\right)\left(p_2 \cosh\phi - E_2 \sinh\phi\right) \\
&= E_2 p_1 \cosh^2\phi - p_1 p_2 \sinh\phi \cosh\phi - E_2 E_1 \cosh\phi \sinh\phi + p_2 E_1 \sinh^2\phi \\
&\quad -\left(E_1 p_2 \cosh^2\phi - p_2 p_1 \sinh\phi \cosh\phi - E_1 E_2 \cosh\phi \sinh\phi + p_1 E_2 \sinh^2\phi\right) \\
&= E_2 p_1 \left(\cosh^2\phi - \sinh^2\phi\right) + E_1 p_2 \left(\sinh^2\phi - \cosh^2\phi\right) = E_2 p_1 - E_1 p_2\,.
\end{aligned}$$

So the proof is complete.