Solutions to Problems 4c

9. A charged pion or kaon decays via $\pi^+ \to \mu^+ \nu_\mu$ or $K^+ \to \mu^+ \nu_\mu$. The Feynman invariant amplitude takes the form $|i\mathcal{M}|^2 = a_\pi^2 \left(p \cdot p' \right)$ or $|i\mathcal{M}|^2 = a_K^2 \left(p \cdot p' \right)$, where a_π and a_K are constants, and *p* and p' are the four-momenta of the final state particles. Calculate the decay rate in terms of *a* and the pion or kaon mass m_π or m_K and muon mass m_μ (a formula for momentum of the final particles was found in problem 2.8b). Treat the anti-neutrino mass as zero. Using the decay rates you found (or I found) from problem 1.7, find the ratio a_K / a_π.

Since we are given the Feynman invariant amplitude, and it is already squared, we can proceed to the next step, which is to replace the momenta with more explicit expressions. If we let *k* be the momentum of the initial pion, then we have

$$k = p + p' ,$$
$$k^2 = p^2 + p'^2 + 2 p \cdot p' ,$$
$$m_\pi^2 = m_\mu^2 + 0 + 2 p \cdot p' .$$

This allows us to easily rewrite the Feynman amplitude for pion decay as

$$|i\mathcal{M}|^2 = \tfrac{1}{2} a_\pi^2 \left(m_\pi^2 - m_\mu^2 \right).$$

Since we are doing a decay into two particles, we use the relevant formulas, which yield

$$\Gamma = \frac{D}{2m_\pi} = \frac{p}{32\pi^2 m_\pi^2} \int |i\mathcal{M}|^2 \, d\Omega = \frac{p}{16\pi m_\pi^2} a_\pi^2 \left(m_\pi^2 - m_\mu^2 \right)$$

The momentum for a massive particle decaying to massless plus massive was worked out in problem 2.8, and is given by $p = \left(m_\pi^2 - m_\mu^2 \right) / 2m_\pi$. We therefore have

$$\Gamma\left(\pi^+ \to \mu^+ \nu_\mu \right) = \frac{a_\pi^2 \left(m_\pi^2 - m_\mu^2 \right)^2}{32\pi m_\pi^3} .$$

It is not hard to see that the formula for the kaon is identical with a few minor tweaks, so

$$\Gamma\left(K^+ \to \mu^+ \nu_\mu \right) = \frac{a_K^2 \left(m_K^2 - m_\mu^2 \right)^2}{32\pi m_K^3} .$$

We then compare the resulting ratio with the experimental value from problem 1.7:

$$\frac{\Gamma\left(\pi^+ \to \mu^+ \nu_\mu \right)}{\Gamma\left(K^+ \to \mu^+ \nu_\mu \right)} = \frac{a_\pi^2 \left(m_\pi^2 - m_\mu^2 \right)^2}{32\pi m_\pi^3} \cdot \frac{32\pi m_K^3}{a_K^2 \left(m_K^2 - m_\mu^2 \right)^2} = \frac{a_\pi^2}{a_K^2} \cdot \frac{m_K^3 \left(m_\pi^2 - m_\mu^2 \right)^2}{m_\pi^3 \left(m_K^2 - m_\mu^2 \right)^2} ,$$

$$\frac{a_K^2}{a_\pi^2} = \frac{\Gamma\left(K^+ \to \mu^+\nu_\mu\right)}{\Gamma\left(\pi^+ \to \mu^+\nu_\mu\right)} \cdot \frac{m_K^3\left(m_\pi^2 - m_\mu^2\right)^2}{m_\pi^3\left(m_K^2 - m_\mu^2\right)^2} = \frac{1}{0.749} \cdot \frac{493.7^3\left(139.6^2 - 105.7^2\right)^2}{139.6^3\left(493.7^2 - 105.7^2\right)^2} = 0.0755\,,$$

$$\frac{a_K}{a_\pi} = \sqrt{0.0755} = 0.275\,.$$

10. The Feynman amplitude for $e^+e^- \to \mu^+\mu^-$, in the high energy limit (energy much larger than the electron or muon mass), is $\left|i\mathcal{M}\right|^2 = e^4\left(1+\cos^2\theta\right)$, where θ is the angle between the initial electron and final muon. Find the total cross section $\sigma\left(e^+e^- \to \mu^+\mu^-\right)$ when they collide head on with energy *E* each. How would your answer change if one had energy *E* and the other energy E'?

This is a two-particle final state cross-section, so we write

$$\sigma = \frac{D}{4\left|E_2\mathbf{p}_1 - E_1\mathbf{p}_2\right|} = \frac{p}{64\pi^2\left|E_2\mathbf{p}_1 - E_1\mathbf{p}_2\right|E_{cm}}\int\left|i\mathcal{M}\right|^2 d\Omega\,.$$

Each of the initial state particles has energy *E*, and since their mass is effectively zero, this is their momenta as well, though they are going in opposite directions. Hence $\left|E_2\mathbf{p}_1 - E_1\mathbf{p}_2\right| = 2E^2$. The center of mass energy is 2*E*, and as we already said, *p* =*E*. Putting this all together, we have

$$\sigma = \frac{E}{64\pi^2 2E^2 2E}\int e^4\left(1+\cos^2\theta\right)d\Omega = \frac{e^4}{256\pi^2E^2}\int_0^{2\pi}d\phi\int_{-1}^{1}\left(1+\cos^2\theta\right)d\cos\theta$$
$$= \frac{e^4}{128\pi E^2}\left(\cos\theta + \tfrac{1}{3}\cos^2\theta\right)\Big|_{\cos\theta=-1}^{\cos\theta=+1} = \frac{e^4}{128\pi E^2}\left(\tfrac{4}{3} - \left(-\tfrac{4}{3}\right)\right) = \frac{e^4}{48\pi E^2}\,.$$

The cross-section is Lorentz invariant, and hence if we write this in terms of *s*, it must be valid in all frames. Since *s* is the square of the center of mass energy, we have $s = \left(2E\right)^2 = 4E^2$, and we have

$$\sigma = \frac{e^4}{12\pi s}\,.$$

Now, as was worked out in a previous problem, if two massless particles collide head on with energies *E* and E', it isn't hard to see that the total energy is $E + E'$ and total momentum is $E - E'$, so that we have

$$s = \left(E + E'\right)^2 - \left(E - E'\right)^2 = 4EE'\,.$$

Substituting this into the previous equation, we therefore have

$$\sigma = \frac{e^4}{48\pi EE'}\,.$$