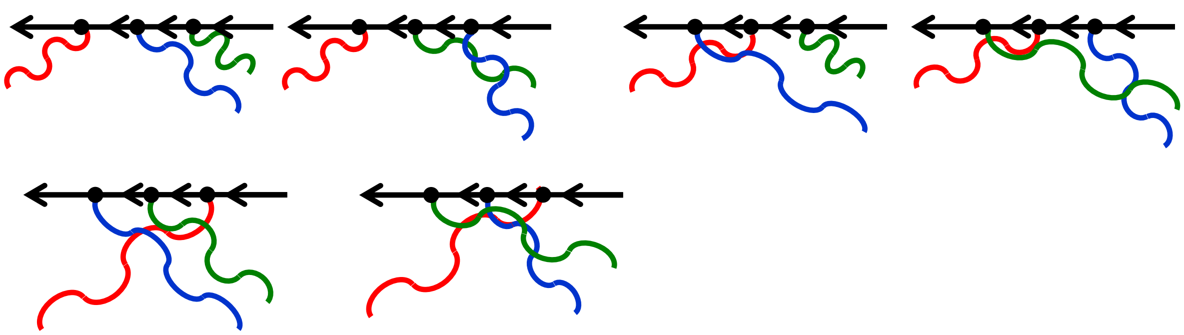
Solutions to Problems 7

3. Draw all six Feynman diagrams for the scattering $e^+\gamma \to e^+\gamma\gamma$ **.**

This is straightforward, but boring. It is more challenging for me because I am not drawing them by hand. I have color coded the photon lines to try to help keep track of which interaction is which.



5. Calculate the total cross section ratio $\sigma\left(e^+e^- \to q\bar{q}\right)/\sigma\left(e^+e^- \to \mu^+\mu^-\right)$**, summed over all quarks (multiply each contribution by 3 for colors), treating the quark as massless if** $\sqrt{s} > 2m$**, and of course ignoring it if** $\sqrt{s} < 2m$**. Do so in the range** $2\text{ GeV} < \sqrt{s} < 3\text{ GeV}$ **(ignore the c quark),** $5\text{ GeV} < \sqrt{s} < 9\text{ GeV}$ **(include the c quark) and** $11\text{ GeV} < \sqrt{s} < 50\text{ GeV}$ **(include the b quark). Compare to the experimental values illustrated in Fig. 7-10 below. What do you think is going on at the other energies shown?**

The requested ratio is just

$$\frac{\sigma\left(e^+e^- \to q\bar{q}\right)}{\sigma\left(e^+e^- \to \mu^+\mu^-\right)} = 3\cdot\frac{4\pi\alpha^2 Q^2}{3s}\cdot\frac{3s}{4\pi\alpha^2\left(-1\right)^2} = 3Q^2 .$$

The contribution from the up or charm quark would be $3\left(\frac{2}{3}\right)^2 = \frac{4}{3}$. The contribution from the down, strange, or bottom quark would be $3\left(-\frac{1}{3}\right)^2 = \frac{1}{3}$. Depending on the energy range, we then have

$$\frac{\sigma\left(e^+e^- \to q\bar{q}\right)}{\sigma\left(e^+e^- \to \mu^+\mu^-\right)} = \left\{\begin{matrix} \frac{4}{3}+2\cdot\frac{1}{3} & 2\text{ GeV} < \sqrt{s} < 3\text{ GeV} \\ 2\cdot\frac{4}{3}+2\cdot\frac{1}{3} & 5\text{ GeV} < \sqrt{s} < 9\text{ GeV} \\ 2\cdot\frac{4}{3}+3\cdot\frac{1}{3} & 11\text{ GeV} < \sqrt{s} < 50\text{ GeV} \end{matrix}\right\} = \begin{cases} 2.00 & 2\text{ GeV} < \sqrt{s} < 3\text{ GeV} \\ 3.33 & 5\text{ GeV} < \sqrt{s} < 9\text{ GeV} \\ 3.67 & 11\text{ GeV} < \sqrt{s} < 50\text{ GeV} \end{cases}$$

I have sketched below dashed lines showing approximately these curve positions. You will note that in the regions listed, they actually work pretty well, though obviously around 40 GeV the *Z*-resonance (which causes weak interactions) is starting to have some effect. As for other energies, below 2 GeV there is clearly something complicated going on (involving the strong interactions of the light quarks), the spikes for the J/ψ and ψ(2s) are caused by resonances (bound states of charm/anti-charm quarks), and there are more resonances around 10 GeV (bound states of bottom-anti bottom quarks).

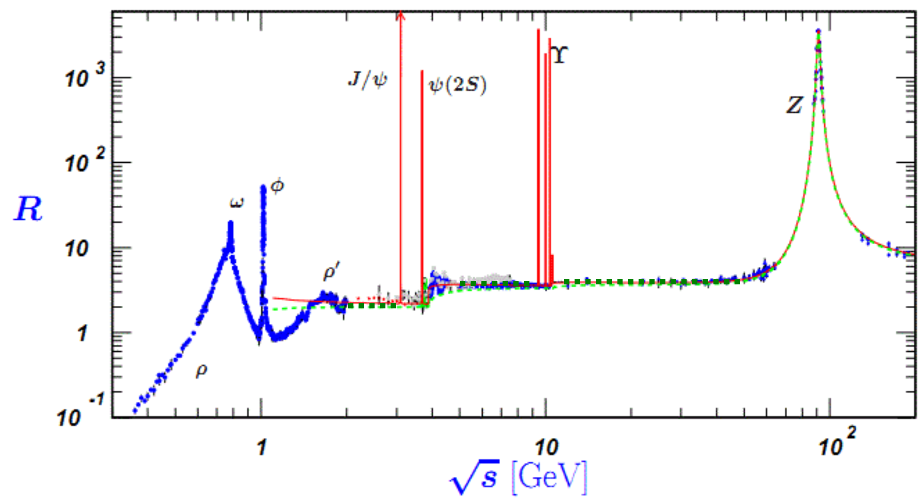


Figure 7-10: The ratio $R = \sigma\left(e^+e^- \to \text{hadrons}\right)/\sigma\left(e^+e^- \to \mu^+\mu^-\right)$. The cross-section to hadrons is believed to be caused by $\sigma\left(e^+e^- \to \text{quarks}\right)$.

6. **Calculate the differential and total cross-section for $e^+e^- \to f\,\bar{f}$, treating the electron mass as zero, but not ignoring the fermion mass *m*. Sketch the result for $3\sigma E^2/\pi Q^2\alpha^2$ as a function of m/E. It should be 1 if *m* = 0.**

We start with eq. (7.19), but we keep the masses. We have

$$\tfrac{1}{4}\sum_{\text{spins}}\left|i\mathcal{M}\right|^2 = \frac{e^4Q^2}{4s^2}\text{Tr}\left(\not{p}_2\gamma^\mu\not{p}_1\gamma^\nu\right)\text{Tr}\left[\left(\not{p}_3+m\right)\gamma_\mu\left(\not{p}_4-m\right)\gamma_\nu\right]$$

$$= \frac{e^4Q^2}{4s^2}\text{Tr}\left(\not{p}_2\gamma^\mu\not{p}_1\gamma^\nu\right)\text{Tr}\left(\not{p}_3\gamma_\mu\not{p}_4\gamma_\nu - m^2\gamma_\mu\gamma_\nu\right)$$

$$= \frac{4e^4Q^2}{s^2}\left(p_2^\mu p_1^\nu + p_1^\mu p_2^\nu - g^{\mu\nu}p_2\cdot p_1\right)\left(p_{3\mu}p_{4\nu} + p_{3\nu}p_{4\mu} - g_{\mu\nu}p_3\cdot p_4 - m^2 g_{\mu\nu}\right)$$

$$= \frac{4e^4Q^2}{s^2}\begin{bmatrix}2\left(p_2\cdot p_3\right)\left(p_1\cdot p_4\right) + 2\left(p_2\cdot p_4\right)\left(p_1\cdot p_3\right) \\ +\left(p_2\cdot p_1\right)\left(p_3\cdot p_4\right)\left(-1-1-1-1+4\right) + m^2\left(p_1\cdot p_2\right)\left(-1-1+4\right)\end{bmatrix}$$

$$= \frac{8e^4Q^2}{s^2}\left[\left(p_2\cdot p_3\right)\left(p_1\cdot p_4\right) + \left(p_2\cdot p_4\right)\left(p_1\cdot p_3\right) + m^2\left(p_1\cdot p_2\right)\right]$$

In the center of mass frame, the electron's energy *E* is the same as their momenta, and we have total energy 2*E*. The final state particles will split this energy between them, and will therefore each have energy *E* as well, but their momenta will be reduced to $p = \sqrt{E^2 - m^2}$. Hence the momenta will be

$$p_1^\mu = \left(E, 0, 0, E\right), \quad p_3^\mu = \left(E, p\sin\theta\cos\phi, p\sin\theta\sin\phi, p\cos\theta\right),$$
$$p_2^\mu = \left(E, 0, 0, -E\right), \quad p_4^\mu = \left(E, -p\sin\theta\cos\phi, -p\sin\theta\sin\phi, -p\cos\theta\right).$$

We then get all the relevant dot products:

$$p_1 \cdot p_4 = p_2 \cdot p_3 = E^2 + Ep\cos\theta\,, \quad p_1 \cdot p_3 = p_2 \cdot p_4 = E^2 - Ep\cos\theta\,, \quad p_1 \cdot p_2 = 2E^2\,.$$

Substituting this in, along with $s = 4E^2$, we have

$$\tfrac{1}{4}\sum_{\text{spins}} \left|i\mathcal{M}\right|^2 = \frac{8e^4Q^2}{\left(4E^2\right)^2}\left[\left(E^2 + Ep\cos\theta\right)^2 + \left(E^2 - Ep\cos\theta\right)^2 + 2m^2E^2\right] = \frac{e^4Q^2}{E^2}\left(E^2 + p^2\cos^2\theta + m^2\right).$$

We now proceed to the cross-section in the usual way.

$$\sigma = \frac{D}{4\left|E_2\mathbf{p}_1 - E_1\mathbf{p}_2\right|} = \frac{1}{8E^2}\frac{p}{16\pi^2E_{cm}}\int d\Omega \frac{1}{4}\sum_{\text{spins}}\left|i\mathcal{M}\right|^2 = \frac{pe^4Q^2}{256\pi^2E^3E^2}\int d\Omega\left(E^2 + p^2\cos^2\theta + m^2\right)$$

$$= \frac{\alpha^2Q^2p}{16E^5}2\pi\int_{-1}^{1}\left(E^2 + p^2\cos^2\theta + m^2\right)d\cos\theta = \frac{\pi\alpha^2Q^2p}{8E^5}\left(2E^2 + \tfrac{2}{3}p^2 + 2m^2\right)$$

$$= \frac{\pi\alpha^2Q^2}{4E^5}\sqrt{E^2 - m^2}\left(E^2 + \tfrac{1}{3}E^2 - \tfrac{1}{3}m^2 + m^2\right) = \frac{\pi Q^2\alpha^2}{3E^2}\left\{\left(1 + \frac{m^2}{2E^2}\right)\sqrt{1 - \frac{m^2}{E^2}}\right\}.$$

The factor in $\{\ \}$'s is what we were asked to graph. A sketch of the result is given at right. Note that for even modestly small values of m/E, it is very close to one; for example, at $m/E = 0.5$ it is already 97.4% of its maximum value.