Physics 744 – Quantum Field Theory

Solution Set 1

1. [15] A set of particles in three dimensions ($\vec r_a = (x_a, y_a, z_a)$) interacts via the Lagrangian

$$L(\vec r_a, \dot{\vec r}_a) = \sum_a \tfrac{1}{2} m_a \dot{\vec r}_a \cdot \dot{\vec r}_a - \sum_a W_a(|\vec r_a|) - \sum_{a<b} V_{ab}(|\vec r_a - \vec r_b|)$$

where W_a and V_{ab} are arbitrary functions of the magnitudes listed. Consider a set of new coordinates

$$\begin{aligned} x'_a &= x_a \cos\theta - y_a \sin\theta \\ y'_a &= y_a \cos\theta + x_a \sin\theta \\ z'_a &= z_a \end{aligned}$$

(a) [10] Show that $L(\vec r'_a, \dot{\vec r}'_a) = L(\vec r_a, \dot{\vec r}_a)$, and that therefore the derivative of the left-hand side with respect to θ is trivial.

We have

$$\begin{aligned} |\vec r'_a| &= \sqrt{(x_a \cos\theta - y_a \sin\theta)^2 + (y_a \cos\theta + x_a \sin\theta)^2 + z_a^2} \\ &= \sqrt{x_a^2 \cos^2\theta - 2x_a y_a \cos\theta \sin\theta + y_a^2 \sin^2\theta + 2x_a y_a \cos\theta \sin\theta + x_a^2 \cos^2\theta + z_a^2} \\ &= \sqrt{x_a^2 + y_a^2 + z_a^2} = |\vec r_a|, \end{aligned}$$

$$\begin{aligned} |\vec r'_a - \vec r'_b| &= \sqrt{(\Delta x \cos\theta - \Delta y \sin\theta)^2 + (\Delta y \cos\theta + \Delta x \sin\theta)^2 + (\Delta z)^2} \\ &= \sqrt{\begin{array}{l}(\Delta x)^2 \cos^2\theta - 2(\Delta x)(\Delta y)\cos\theta \sin\theta + (\Delta y)^2 \sin^2\theta \\ +2(\Delta x)(\Delta y)\cos\theta\sin\theta + (\Delta x)^2\cos^2\theta + (\Delta z)^2\end{array}} \\ &= \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2} = |\vec r_a - \vec r_b|, \end{aligned}$$

$$\begin{aligned} \dot{\vec r}'_a \cdot \dot{\vec r}'_a &= (\dot x_a \cos\theta - \dot y_a \sin\theta)^2 + (\dot y_a \cos\theta + \dot x_a \sin\theta)^2 + \dot z_a^2 \\ &= \dot x_a^2 \cos^2\theta - 2\dot x_a \dot y_a \cos\theta \sin\theta + \dot y_a^2 \sin^2\theta \\ &\quad + \dot y_a^2 \cos^2\theta + 2\dot x_a \dot y_a \cos\theta\sin\theta + \dot x_a^2 \sin^2\theta + \dot z_a^2 \\ &= \dot x_a^2 + \dot y_a^2 + \dot z_a^2 = \dot{\vec r}_a \cdot \dot{\vec r}_a . \end{aligned}$$

Thus it is obvious that everything in the Lagrangian is unchanged, and hence the derivative of the Lagrangian with respect to θ is trivial.

(b) [5] Deduce the corresponding conserved quantity, and identify it.

The corresponding conserved quantity is

$$Q=\sum_a\left\{\frac{\partial L}{\partial\dot{x}_a}\frac{\partial x'_a}{\partial\theta}\bigg|_{\theta=0}+\frac{\partial L}{\partial\dot{y}_a}\frac{\partial y'_a}{\partial\theta}\bigg|_{\theta=0}+\frac{\partial L}{\partial\dot{z}_a}\frac{\partial z'_a}{\partial\theta}\bigg|_{\theta=0}\right\}=\sum_a\left\{m_a\dot{x}_a\left(-y_a\right)+m_a\dot{y}_a x_a+m_a\dot{z}_a 0\right\}$$

$$=\sum_a\left(x_a p_{ya}-y_a p_{xa}\right)=\sum_a L_{za}=L_z$$

It's just the z-component of the angular momentum.

2. **[10] A set of particles in three dimensions ($\vec{r}_a = (x_a, y_a, z_a)$) interacts via the Lagrangian**

$$L(\vec{r}_a, \dot{\vec{r}}_a) = \sum_a \tfrac{1}{2} m_a \dot{\vec{r}}_a \cdot \dot{\vec{r}}_a - \sum_{a<b} V_{ab}(\vec{r}_a - \vec{r}_b)$$

where the V_{ab} are arbitrary functions of the differences of the coordinates. Consider the Galilean transformation

$$\begin{aligned} x'_a &= x_a + vt \\ y'_a &= y_a \\ z'_a &= z_a \end{aligned}$$

(a) [6] Calculate $L(\vec{r}'_a, \dot{\vec{r}}'_a)$, and show that although its derivative with respect to v at v = 0 is non-zero, it can be written as a total time derivative of some quantity.

We have

$$\begin{aligned} L(\vec{r}'_a, \dot{\vec{r}}'_a) &= \tfrac{1}{2}\sum_a m_a \left[\frac{d}{dt}(\vec{r}_a + \hat{x}vt)\right]^2 - \sum_{a<b} V_{ab}(\vec{r}_a + \hat{x}vt - \vec{r}_b - \hat{x}vt) \\ &= \sum_a \tfrac{1}{2} m_a \dot{\vec{r}}_a \cdot \dot{\vec{r}}_a + v\sum_a m_a v_{ax} + \sum_a \tfrac{1}{2} m_a v^2 - \sum_{a<b} V_{ab}(\vec{r}_a - \vec{r}_b) \\ &= L(\vec{r}_a, \dot{\vec{r}}_a) + v\sum_a m_a \dot{x}_a + \tfrac{1}{2} v^2 \sum_a m_a \end{aligned}$$

Taking the derivative, we have

$$\frac{d}{dv} L(\vec{r}'_a, \dot{\vec{r}}'_a)\Big|_{v=0} = \sum_a m_a \dot{x}_a = \frac{d}{dt}\sum_a m_a x_a$$

(b) [4] Find a quantity which is consequently conserved; that is, whose time derivative is zero. Write this quantity in terms of the total mass M, some part of the total momentum $\vec{P} = (P_x, P_y, P_z)$, and the center of mass coordinate $\vec{R} = (X, Y, Z)$.

The resulting conserved quantity is therefore

$$\begin{aligned} Q &= \sum_a \left\{ \frac{\partial L}{\partial \dot{x}_a} \frac{\partial x'_a}{\partial v}\Big|_{v=0} + \frac{\partial L}{\partial \dot{y}_a} \frac{\partial y'_a}{\partial v}\Big|_{v=0} + \frac{\partial L}{\partial \dot{z}_a} \frac{\partial z'_a}{\partial v}\Big|_{v=0} - m_a x_a \right\} \\ &= \sum_a \{ m_a \dot{x}_a t + 0 + 0 - m_a x_a \} = P_x t - MX \end{aligned}$$

The last term is the center of mass coordinate, since $\vec{R} = M^{-1}\sum_a m_a \vec{r}_a$.

3. Two particles are moving in one dimension with Lagrangian

$$L = \tfrac{1}{2} M\dot{x}_1^2 + 2M\dot{x}_2^2 - \tfrac{1}{2} M\omega^2 \left[x_1^2 + 4x_1x_2 + 10x_2^2 \right]$$

(a) Find a change of variables $x_1, x_2 \to y_1, y_2$ so that the Lagrangian, rewritten in terms of the *y*'s, takes the form

$$L = \tfrac{1}{2} M \left(\dot{y}_1^2 + \dot{y}_2^2 \right) - \tfrac{1}{2} K_{ij} y_i y_j$$

what is the matrix *K*? Note that K_{ij} must be symmetric, so the coefficient of y_1y_2 must be cut in half to deduce $K_{12} = K_{21}$.

This is straightforward. It is obvious that this works if you define

$$y_1 = x_1, \quad y_2 = 2x_2 .$$

In terms of these, the Lagrangian is just

$$L = \tfrac{1}{2} M \left(\dot{y}_1^2 + \dot{y}_2^2 \right) - \tfrac{1}{2} M\omega^2 \left[y_1^2 + 2y_1y_2 + \tfrac{5}{2} y_2^2 \right]$$

From this we can read off the matrix *K*:

$$K = M\omega^2 \begin{pmatrix} 1 & 1 \\ 1 & \tfrac{5}{2} \end{pmatrix}$$

(b) Find the eigenvalues of *K* and the corresponding orthonormal eigenvectors. If the eigenvalues of *K* are complicated, then either you or I have made a mistake.

To find the eigenvalues, we subtract $\lambda\mathbf{1}$ from the matrix and set the resulting determinant to zero. We therefore have

$$0 = \det \begin{pmatrix} 1-\lambda & 1 \\ 1 & \tfrac{5}{2}-\lambda \end{pmatrix} = (1-\lambda)\left(\tfrac{5}{2}-\lambda\right) - 1 = \lambda^2 - \tfrac{7}{2}\lambda + \tfrac{3}{2},$$

$$\lambda = \frac{\tfrac{7}{2} \pm \sqrt{\tfrac{49}{4} - 4\cdot\tfrac{3}{2}}}{2} = \frac{\tfrac{7}{2} \pm \sqrt{\tfrac{25}{4}}}{2} = \frac{7 \pm 5}{4} = 3 \text{ or } \tfrac{1}{2}$$

The eigenvalues of *K* are therefore $3M\omega^2$ and $\tfrac{1}{2}M\omega^2$. To find the eigenvectors, we need to solve the equations

$$\begin{pmatrix} 1 & 1 \\ 1 & \tfrac{5}{2} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 3 \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 1 & \tfrac{5}{2} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

These equations, written explicitly, work out to

$$\left\{ \begin{matrix} \alpha + \beta = 3\alpha \\ \alpha + \tfrac{5}{2}\beta = 3\beta \end{matrix} \right\} \quad \text{and} \quad \left\{ \begin{matrix} \alpha + \beta = \tfrac{1}{2}\alpha \\ \alpha + \tfrac{5}{2}\beta = \tfrac{1}{2}\beta \end{matrix} \right\}$$

We need to solve the left set of equations or the right set of equations to find each of the eigenvectors. In each case, the equations simplify so that you end up with only a single equation, namely

$$\beta = 2\alpha \quad \text{and} \quad \alpha = -2\beta$$

This tells us the corresponding eigenvectors are

$$\alpha \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \beta \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

We want these to be normalized, so in summary our eigenvectors are

$$\left| 3M\omega^2 \right\rangle = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \left| \tfrac{1}{2} M\omega^2 \right\rangle = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

(c) Find a change of variables $y_1, y_2 \to z_1, z_2$ such that the Lagrangian now takes the form

$$L = \tfrac{1}{2} M \left(\dot{z}_1^2 + \dot{z}_2^2 \right) - \tfrac{1}{2} k_1 z_1^2 - \tfrac{1}{2} k_2 z_2^2$$

The eigenvectors tell us the change of variables we need, namely

$$z_1 = \tfrac{1}{\sqrt{5}} \left(y_1 + 2y_2 \right) \quad \text{and} \quad z_2 = \tfrac{1}{\sqrt{5}} \left(y_2 - 2y_1 \right)$$

It's then pretty easy to see that

$$\begin{aligned} 3M\omega^2 z_1^2 + \tfrac{1}{2} M\omega^2 z_2^2 &= M\omega^2 \left[\tfrac{3}{5} \left(y_1 + 2y_2 \right)^2 + \tfrac{1}{10} \left(y_2 - 2y_1 \right)^2 \right] \\ &= M\omega^2 \left[\tfrac{3}{5} y_1^2 + \tfrac{12}{5} y_1 y_2 + \tfrac{12}{5} y_2^2 + \tfrac{1}{10} y_2^2 - \tfrac{2}{5} y_1 y_2 + \tfrac{2}{5} y_1^2 \right] \\ &= M\omega^2 \left[y_1^2 + 2y_1 y_2 + \tfrac{5}{2} y_2^2 \right] \\ \dot{z}_1^2 + \dot{z}_2^2 &= \tfrac{1}{5} \left[\left(\dot{y}_1 + 2\dot{y}_2 \right)^2 + \left(\dot{y}_2 - 2\dot{y}_1 \right)^2 \right] \\ &= \tfrac{1}{5} \dot{y}_1^2 + \tfrac{4}{5} \dot{y}_1 \dot{y}_2 + \tfrac{4}{5} \dot{y}_2^2 + \tfrac{1}{5} \dot{y}_2^2 - \tfrac{4}{5} \dot{y}_1 \dot{y}_2 + \tfrac{4}{5} \dot{y}_1^2 = \dot{y}_1^2 + \dot{y}_2^2 \end{aligned}$$

Substituting this into the equations above, we have

$$L = \tfrac{1}{2} M \left(\dot{z}_1^2 + \dot{z}_2^2 \right) - \tfrac{1}{2} \left[3M\omega^2 z_1^2 + \tfrac{1}{2} M\omega^2 z_2^2 \right]$$

(d) What are the normal frequencies of this system?

The frequencies are given by $\sqrt{k_i / M}$, which work out to $\omega\sqrt{3}$ and $\omega / \sqrt{2}$.