Physics 744 – Quantum Field Theory

Solution Set 3

1. [10] Consider a Lagrangian for two scalar fields ϕ_1 and ϕ_2, which has no more than two derivatives, and is no higher than quadratic order in the fields. The Lagrangian must be of the form

$$\mathcal{L} = \tfrac{1}{2} A\partial_\mu\phi_1\partial^\mu\phi_1 + B\partial_\mu\phi_1\partial^\mu\phi_2 + \tfrac{1}{2} C\partial_\mu\phi_2\partial^\mu\phi_2 - V\left(\phi_1,\phi_2\right)$$

(a) [5] Define ϕ_1' by the equation $\phi_1 = \phi_1' - B\phi_2/A$. Show that the kinetic term in $\mathcal{L}$ when rewritten in terms of ϕ_1' and ϕ_2 have the exact same form, except that B = 0. So without loss of generality, we can assume B = 0.

Obviously, we can ignore the potential term. Focusing exclusively on the kinetic terms, we would have

$$\begin{aligned}\mathcal{L} &= \tfrac{1}{2} A\partial_\mu\left(\phi_1' - B\phi_2/A\right)\partial^\mu\left(\phi_1' - B\phi_2/A\right) + B\partial_\mu\left(\phi_1' - B\phi_2/A\right)\partial^\mu\phi_2 + \tfrac{1}{2} C\partial_\mu\phi_2\partial^\mu\phi_2 \\ &\quad - V\left(\phi_1' - B\phi_2/A, \phi_2\right) \\ &= \tfrac{1}{2} A\partial_\mu\phi_1'\partial^\mu\phi_1' - B\partial_\mu\phi_1'\partial^\mu\phi_2 + \tfrac{1}{2} B^2\,\partial_\mu\phi_2\partial^\mu\phi_2/A + B\partial_\mu\phi_1'\partial^\mu\phi - B^2\,\partial_\mu\phi_2\partial^\mu\phi_2/A \\ &\quad + \tfrac{1}{2} C\partial_\mu\phi_2\partial^\mu\phi_2 - V\left(\phi_1' - B\phi_2/A, \phi_2\right) \\ &= \tfrac{1}{2} A\partial_\mu\phi_1'\partial^\mu\phi_1' + \tfrac{1}{2}\left(C - B^2/A\right)\partial_\mu\phi_2\partial^\mu\phi_2 - V\left(\phi_1' - B\phi_2/A, \phi_2\right)\end{aligned}$$

Qualitatively, this is the same thing we started with, but the B term removed. So we need never consider Lagrangians with cross-terms for the kinetic terms. We'll drop the primes and assume B = 0.

(b) [5] Now work out the Hamiltonian for this system. Argue that it is bounded below (never very negative) only if A and C are both positive and V is also bounded below. Argue that by rescaling the two fields, we can always make $A = C = +1$.

Finding the Hamiltonian is straightforward. We have

$$\mathcal{H} = \frac{\partial\mathcal{L}}{\partial\dot\phi_1}\dot\phi_1 + \frac{\partial\mathcal{L}}{\partial\dot\phi_2}\dot\phi_2 - \mathcal{L} = \tfrac{1}{2} A\left[\dot\phi_1^2 + \left(\nabla\phi_1\right)^2\right] + \tfrac{1}{2} C\left[\dot\phi_2^2 + \left(\nabla\phi_2\right)^2\right] - V\left(\phi_1,\phi_2\right)$$

The kinetic terms are positive definite, until they are multiplied by A and C. If A or C were negative, we could make this as negative as we want simply by having the corresponding field oscillate rapidly in space or time. To avoid this catastrophe, demand that A and C are positive. Then if we define

$$\phi_1' = \phi_1\sqrt{A}, \quad \phi_2' = \phi_2\sqrt{C},$$

Then it is easy to see that the Lagrangian in terms of these new fields is just

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi_1'\partial^\mu\phi_1' + \tfrac{1}{2}\partial_\mu\phi_2'\partial^\mu\phi_2' - V\left(\phi_1'/\sqrt{A}, \phi_2'/\sqrt{C}\right)$$

Hence we can assume, without loss of generality, that $A = C = 1$.

2. [15] The Lagrangian of the previous problem has now been reduced to the form

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \tfrac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - V\left(\phi_1, \phi_2\right)$$

(a) [4] Since the potential is no higher than quadratic, it must be of the form

$$V\left(\phi_1, \phi_2\right) = D + E\phi_1 + F\phi_2 + \tfrac{1}{2}A\phi_1^2 + B\phi_1\phi_2 + \tfrac{1}{2}C\phi_2^2$$

Since this potential is bounded below, it must have a minimum somewhere. Argue that if we shift ϕ_1 and ϕ_2 by adding constants to them, so that the new minimum is at $\phi_1 = 0 = \phi_2$, two of the terms will automatically vanish. Also, explain why the *D* term is irrelevant.

The Hamiltonian density will be given as above. To be bounded below, the potential must therefore be bounded below, which can only happen if it is constant or has a minimum. If you shift to the minimum. then the derivative with respect to either of the fields must vanish, so $\partial V/\partial\phi_1 = 0 = \partial V/\partial\phi_2$ at the origin. It follows that $E = F = 0$.

Also, looking at the Euler-Lagrange equations, it is obvious that a constant will make no difference, so without loss of generality, we can assume $D = 0$.

(b) [5] Consider the field transformation

$$\phi_1 = \phi_1'\cos\theta - \phi_2'\sin\theta$$
$$\phi_2 = \phi_1'\sin\theta + \phi_2'\cos\theta$$

Convince yourself (and me) that the kinetic term is unchanged by this field transformation.

The kinetic term will be given by

$$\begin{aligned}
\partial_\mu\phi_1\partial^\mu\phi_1 + \partial_\mu\phi_2\partial^\mu\phi_2 &= \left(\cos\theta\partial_\mu\phi_1' - \sin\theta\partial_\mu\phi_2'\right)\left(\cos\theta\partial^\mu\phi_1' - \sin\theta\partial^\mu\phi_2'\right) \\
&\quad + \left(\sin\theta\partial_\mu\phi_1' + \cos\theta\partial_\mu\phi_2'\right)\left(\sin\theta\partial^\mu\phi_1' + \cos\theta\partial^\mu\phi_2'\right) \\
&= \partial_\mu\phi_1'\partial^\mu\phi_1'\begin{pmatrix}\cos^2\theta \\ +\sin^2\theta\end{pmatrix} + \partial_\mu\phi_1'\partial^\mu\phi_2'\begin{pmatrix}-2\sin\theta\cos\theta \\ +2\sin\theta\cos\theta\end{pmatrix} + \partial_\mu\phi_2'\partial^\mu\phi_2'\begin{pmatrix}\sin^2\theta \\ +\cos^2\theta\end{pmatrix} \\
&= \partial_\mu\phi_1'\partial^\mu\phi_1' + \partial_\mu\phi_2'\partial^\mu\phi_2'
\end{aligned}$$

Hence the kinetic term is clearly unchanged.

(c) [6] Show that the same change of field definitions can simplify the potential. Specifically, show that we can make *B* vanish if we choose

$$\tan(2\theta) = \frac{2B}{A-C}$$

Making the same substitutions into the potential, we have

$$\begin{aligned}
V &= \tfrac{1}{2} A\phi_1^2 + B\phi_1\phi_2 + \tfrac{1}{2} C\phi_2^2 \\
&= \tfrac{1}{2} A\left(\phi_1' \cos\theta - \phi_2' \sin\theta\right)^2 + B\left(\phi_1' \cos\theta - \phi_2' \sin\theta\right)\left(\phi_1' \sin\theta + \phi_2' \cos\theta\right) \\
&\quad + \tfrac{1}{2} C\left(\phi_1' \sin\theta + \phi_2' \cos\theta\right)^2 \\
&= \phi_1'^2 \left(\tfrac{1}{2} A\cos^2\theta + B\cos\theta\sin\theta + \tfrac{1}{2} C\sin^2\theta\right) \\
&\quad + \phi_1'\phi_2' \left(-A\sin\theta\cos\theta + B\cos^2\theta - B\sin^2\theta + C\sin\theta\cos\theta\right) \\
&\quad + \phi_2'^2 \left(\tfrac{1}{2} A\sin^2\theta - B\sin\theta\cos\theta + \tfrac{1}{2} C\cos^2\theta\right)
\end{aligned}$$

The middle term will vanish if we have

$$\begin{aligned}
B\left(\cos^2\theta - \sin^2\theta\right) &= \sin\theta\cos\theta\left(C-A\right), \\
B\cos(2\theta) &= \tfrac{1}{2}\left(C-A\right)\sin(2\theta), \\
\tan(2\theta) &= \frac{2B}{C-A}.
\end{aligned}$$

Hence we can make the term *B* vanish. Hence the theory, despite its apparent initial complexity, actually has only two parameters, *A* and *C*. The square roots of these will be the masses of the two particles.

3. [15] Two fields ϕ_1 and ϕ_2 interact via a Lagrangian of the form

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \tfrac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - V\left(\phi_1^2+\phi_2^2\right)$$

where V is an arbitrary function. We will be considering the symmetry

$$\phi_1' = \phi_1\cos\theta - \phi_2\sin\theta,$$
$$\phi_2' = \phi_1\sin\theta + \phi_2\cos\theta.$$

(a) [3] Show that this is a symmetry, and the Lagrangian is unchanged by this transformation. (technically, you should also check that $\theta = 0$ is the null transformation).

We note that in the previous problem, part (b), we showed the kinetic term is unchanged. We can also use the results of part (c) with $A = C = 2$ and $B = 0$ to see that

$$\begin{aligned}\phi_1^2+\phi_2^2 &= \phi_1'^2\left(\cos^2\theta+\sin^2\theta\right)+\phi_1'\phi_2'\left(-2\sin\theta\cos\theta+2\sin\theta\cos\theta\right)+\phi_2'^2\left(\sin^2\theta+\cos^2\theta\right)\\ &= \phi_1'^2+\phi_2'^2\end{aligned}$$

It is therefore clear that the potential term is also conserved.

(b) [2] Derive an expression for the corresponding conserved current.

The conserved current is given by

$$J^\mu = \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi_1\right)}\frac{\partial\phi_1'}{\partial\theta}\bigg|_{\theta=0} + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi_2\right)}\frac{\partial\phi_2'}{\partial\theta}\bigg|_{\theta=0} = -\phi_2\partial^\mu\phi_1 + \phi_1\partial^\mu\phi_2$$

(c) [4] Let ϕ be the complex field defined by $\phi = \left(\phi_1 + i\phi_2\right)/\sqrt{2}$. Rewrite the Lagrangian density $\mathcal{L}$ in terms of ϕ and ϕ^*.

It isn't hard to see that

$$\partial_\mu\phi\partial^\mu\phi^* = \tfrac{1}{2}\left(\partial_\mu\phi_1 + i\partial_\mu\phi_2\right)\left(\partial^\mu\phi_1 - i\partial^\mu\phi_2\right) = \tfrac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \tfrac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2,$$
$$\phi\phi^* = \tfrac{1}{2}\left(\phi_1 + i\phi_2\right)\left(\phi_1 - i\phi_2\right) = \tfrac{1}{2}\phi_1^2 + \tfrac{1}{2}\phi_2^2$$

It follows that

$$\mathcal{L} = \partial_\mu\phi\partial^\mu\phi^* - V\left(2\phi\phi^*\right).$$

This simpler form means we can often write things more simply in terms of these complex fields than the standard real fields.

(d) [3] Rewrite the transformation above in the form $\phi'(x) = f(\theta)\phi(x)$. What is the function *f*? Verify directly that the transformation leaves $\mathcal{L}$ unchanged in terms of this notation.

We see that

$$\phi' = \phi_1' + i\phi_2' = (\phi_1\cos\theta - \phi_2\sin\theta) + i(\phi_2\cos\theta + \phi_1\sin\theta) = \cos\theta(\phi_1 + i\phi_2) + \sin\theta(-\phi_2 + i\phi_1)$$
$$= (\phi_1 + i\phi_2)(\cos\theta + i\sin\theta) = e^{i\theta}\phi$$

That this is a symmetry of the Lagrangian is obvious:

$$\mathcal{L}' = \partial_\mu\left(e^{i\theta}\phi\right)\partial^\mu\left(e^{-i\theta}\phi^*\right) - V\left(2e^{i\theta}\phi e^{-i\theta}\phi^*\right) = \partial_\mu\phi\partial^\mu\phi^* - V\left(2\phi\phi^*\right) = \mathcal{L}$$

(e) [3] A naïve expression for the current density in terms of ϕ would be

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\frac{\partial\phi'}{\partial\theta}\bigg|_{\theta=0} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\frac{\partial\phi'^*}{\partial\theta}\bigg|_{\theta=0}$$

Show that this naïve expectation is correct; that is, it leads to exactly the same current you found in part (b).

Using the naïve formula, we find

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}i\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\left(-i\phi^*\right) = i\phi\partial^\mu\phi^* - i\phi^*\partial^\mu\phi$$
$$= \tfrac{1}{2}i\left[(\phi_1 + i\phi_2)\partial^\mu(\phi_1 - i\phi_2) - (\phi_1 - i\phi_2)\partial^\mu(\phi_1 + i\phi_2)\right] = \tfrac{1}{2}i\left[2i\phi_2\partial^\mu\phi_1 - 2i\phi_1\partial^\mu\phi_2\right]$$
$$= \phi_1\partial^\mu\phi_2 - \phi_2\partial^\mu\phi_1 .$$

This is exactly what we found before.

4. [10] A real field has Lagrangian

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi + \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{24}\gamma\phi^4$$

Consider the "scale invariance" symmetry, $\phi'(\mathbf{x},\lambda) = e^\lambda\phi(\mathbf{x}e^\lambda)$

(a) [4] Convince yourself (and me) that

$$\left.\frac{d}{d\lambda}\phi'(\mathbf{x},\lambda)\right|_{\lambda=0} = \phi(\mathbf{x}) + x^\nu\partial_\nu\phi(\mathbf{x}) \quad \text{and} \quad \partial_\mu\left.\frac{d}{d\lambda}\phi'(\mathbf{x},\lambda)\right|_{\lambda=0} = 2\partial_\mu\phi(\mathbf{x}) + x^\nu\partial_\nu\partial_\mu\phi(\mathbf{x})$$

We have

$$\begin{aligned}\left.\frac{d}{d\lambda}\phi'(\mathbf{x},\lambda)\right|_{\lambda=0} &= e^\lambda\phi(e^\lambda\mathbf{x})\Big|_{\lambda=0} + e^\lambda\frac{d}{d\lambda}\phi(e^\lambda\mathbf{x})\Big|_{\lambda=0} = \phi(\mathbf{x}) + e^\lambda\partial_\mu\phi(e^\lambda\mathbf{x})e^\lambda x^\mu\Big|_{\lambda=0} \\ &= \phi(\mathbf{x}) + x^\mu\partial_\mu\phi(\mathbf{x}),\end{aligned}$$

Taking the derivative of this, we have

$$\begin{aligned}\partial_\mu\left.\frac{d}{d\lambda}\phi'(\mathbf{x},\lambda)\right|_{\lambda=0} &= \partial_\mu\left[\phi(\mathbf{x}) + x^\nu\partial_\nu\phi(\mathbf{x})\right] = \partial_\mu\phi(\mathbf{x}) + (\partial_\mu x^\nu)\partial_\nu\phi(\mathbf{x}) + x^\nu\partial_\mu\partial_\nu\phi(\mathbf{x}) \\ &= \partial_\mu\phi(\mathbf{x}) + \delta_\mu^\nu\partial_\nu\phi(\mathbf{x}) + x^\nu\partial_\mu\partial_\nu\phi(\mathbf{x}) = 2\partial_\mu\phi(\mathbf{x}) + x^\nu\partial_\mu\partial_\nu\phi(\mathbf{x})\end{aligned}$$

(b) [6] Show that

$$\left.\frac{d}{d\lambda}\mathcal{L}(\phi',\partial_\mu\phi')\right|_{\lambda=0} = 4\mathcal{L} + x^\nu\partial_\nu\mathcal{L} = \partial_\nu(x^\nu\mathcal{L})$$

but only if one of the terms in the Lagrangian vanishes. Hence this is a symmetry only if one of the terms is zero. Which one? (note that this statement is only true in four space-time dimensions).

We have

$$\begin{aligned}\left.\frac{d}{d\lambda}\mathcal{L}(\phi',\partial_\mu\phi')\right|_{\lambda=0} &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\left(2\partial_\mu\phi + x^\nu\partial_\mu\partial_\nu\phi\right) + \frac{\partial\mathcal{L}}{\partial\phi}\left(\phi + x^\mu\partial_\mu\phi\right) \\ &= \partial^\mu\phi\left(2\partial_\mu\phi + x^\nu\partial_\mu\partial_\nu\phi\right) - \left(m^2\phi + \tfrac{1}{6}\gamma\phi^3\right)\left(\phi + x^\mu\partial_\mu\phi\right) \\ &= 2\partial^\mu\phi\partial_\mu\phi - m^2\phi^2 - \tfrac{1}{6}\gamma\phi^4 + x^\nu\partial^\mu\phi\partial_\nu\partial_\mu\phi - m^2x^\mu\phi\partial_\mu\phi - \tfrac{1}{6}\gamma x^\mu\phi^3\partial_\mu\phi\end{aligned}$$

By comparison,

$$4\mathcal{L} + x^\nu\partial_\nu\mathcal{L} = 2\partial_\mu\phi\partial^\mu\phi - 2m^2\phi^2 - \tfrac{1}{6}\gamma\phi^4 + x^\nu\partial^\mu\phi\partial_\nu\partial_\mu\phi - m^2x^\nu\phi\partial_\nu\phi - \tfrac{1}{6}\gamma x^\nu\phi^3\partial_\nu\phi.$$

Comparing these expressions, we see that they match only if $m^2 = 0$. Hence this theory is scale invariant only if the field is massless. We then prove the remaining identity by noting that $\partial_\nu(x^\nu\mathcal{L}) = \partial_\nu x^\nu\mathcal{L} + x^\nu\partial_\nu\mathcal{L} = \delta_\nu^\nu\mathcal{L} + x^\nu\partial_\nu\mathcal{L} = 4\mathcal{L} + x^\nu\partial_\nu\mathcal{L}$.