Solution Set 4

1. [20] A single real scalar field has the usual Lagrangain, $\mathcal{L}=\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-\frac{1}{2}m^2\phi^2$**.**

(a) [3] Work out the components of T^{0i}**, three components of the stress-energy tensor. Write it in terms of** $\phi(\mathbf{x})$**,** $\nabla\phi(\mathbf{x})$**, and** $\pi(\mathbf{x})$**. Also, write an expression for** $\vec{P}$**, the three-momentum.**

The components of T^{0i} are given by

$$T^{0i}=\partial^0\phi\partial^i\phi-g^{0i}\mathcal{L}=-\partial_0\phi\partial_i\phi=-\pi\partial_i\phi$$

The three momentum, therefore, is

$$\vec{P}=\int T^{0i}d^3\vec{x}=-\int\pi(\mathbf{x})\nabla\phi(\mathbf{x})\,d^3\vec{x}$$

(b) [12] Substitute the expression for $\phi(\mathbf{x})$ **and** $\pi(\mathbf{x})$ **in terms of creation and annihilation operators into this expression, and hence find a simple expression for** $\vec{P}$ **in terms of creation and annihilation operators** $\alpha_{\vec{k}}^\dagger$ **and** $\alpha_{\vec{k}}$**.**

We will work in the Heisenberg representation. In this representation, we have

$$\begin{aligned}
\vec{P}&=-\int\pi(\mathbf{x})\nabla\phi(\mathbf{x})\,d^3\vec{x}\\
&=-\int d^3\vec{x}\int\frac{d^3\vec{k}}{(2\pi)^3 2\omega_k}\int\frac{d^3\vec{k}'}{(2\pi)^3 2\omega_{k'}}i\omega_k\left[-e^{i\vec{k}\cdot\vec{x}-i\omega_k t}\alpha_{\vec{k}}+e^{-i\vec{k}\cdot\vec{x}+i\omega_k t}\alpha_{\vec{k}}^\dagger\right]\\
&\quad\nabla\left[e^{i\vec{k}'\cdot\vec{x}-i\omega_{k'}t}\alpha_{\vec{k}'}+e^{-i\vec{k}'\cdot\vec{x}+i\omega_{k'}t}\alpha_{\vec{k}'}^\dagger\right]\\
&=\int\frac{d^3\vec{k}}{(2\pi)^3 2}\int\frac{d^3\vec{k}'}{(2\pi)^3 2\omega_{k'}}\vec{k}'\int d^3\vec{x}\left[e^{i\vec{k}\cdot\vec{x}-i\omega_k t}\alpha_{\vec{k}}-e^{-i\vec{k}\cdot\vec{x}+i\omega_k t}\alpha_{\vec{k}}^\dagger\right]\left[-e^{i\vec{k}'\cdot\vec{x}-i\omega_{k'}t}\alpha_{\vec{k}'}+e^{-i\vec{k}'\cdot\vec{x}+i\omega_{k'}t}\alpha_{\vec{k}'}^\dagger\right]\\
&=\int\frac{d^3\vec{k}}{(2\pi)^3 2}\int\frac{d^3\vec{k}'}{(2\pi)^3 2\omega_{k'}}\vec{k}'(2\pi)^3\left[\begin{array}{l}\delta^3(\vec{k}-\vec{k}')\left(\alpha_{\vec{k}}\alpha_{\vec{k}'}^\dagger e^{-i\omega_k t+i\omega_{k'}t}+\alpha_{\vec{k}}^\dagger\alpha_{\vec{k}'}e^{i\omega_k t-i\omega_{k'}t}\right)\\+\delta^3(\vec{k}+\vec{k}')\left(\alpha_{\vec{k}}\alpha_{\vec{k}'}e^{-i\omega_k t-i\omega_{k'}t}+\alpha_{\vec{k}}^\dagger\alpha_{\vec{k}'}^\dagger e^{i\omega_k t+i\omega_{k'}t}\right)\end{array}\right]\\
&=\int\frac{d^3\vec{k}}{(2\pi)^3 4\omega_k}\vec{k}\left[\alpha_{\vec{k}}\alpha_{\vec{k}}^\dagger+\alpha_{\vec{k}}^\dagger\alpha_{\vec{k}}+\alpha_{\vec{k}}\alpha_{-\vec{k}}e^{-2i\omega_k t}+\alpha_{\vec{k}}^\dagger\alpha_{-\vec{k}}^\dagger e^{2i\omega_k t}\right]
\end{aligned}$$

The first two terms don't look so bad, but the latter two don't look nice. To make it look nicer, split either of the last two terms in half, and then make the substitution $\vec{k}\to-\vec{k}$ on these terms. This will reduce the expression to

$$\vec{P}=\int\frac{d^3\vec{k}}{(2\pi)^3\,4\omega_k}\left[\begin{array}{l}\vec{k}\alpha_{\vec{k}}\alpha_{\vec{k}}^\dagger+\vec{k}\alpha_{\vec{k}}^\dagger\alpha_{\vec{k}}+\frac{1}{2}\vec{k}\alpha_{\vec{k}}\alpha_{-\vec{k}}e^{-2i\omega_k t}-\frac{1}{2}\vec{k}\alpha_{-\vec{k}}\alpha_{\vec{k}}e^{-2i\omega_k t}\\+\frac{1}{2}\vec{k}\alpha_{\vec{k}}^\dagger\alpha_{-\vec{k}}^\dagger e^{2i\omega_k t}-\frac{1}{2}\vec{k}\alpha_{-\vec{k}}^\dagger\alpha_{\vec{k}}^\dagger e^{2i\omega_k t}\end{array}\right]$$

It is now obvious from the commutation relations that all but the first two terms cancel. The only remaining complication is the term $\alpha_{\vec{k}}\alpha_{\vec{k}}^\dagger$, which we can commute into the usual order to yield $\alpha_{\vec{k}}\alpha_{\vec{k}}^\dagger=\alpha_{\vec{k}}^\dagger\alpha_{\vec{k}}+\left[\alpha_{\vec{k}},\alpha_{\vec{k}}^\dagger\right]=(2\omega_k)(2\pi)^3\delta^3(0)$. If we take this at face value, it yields

$$\vec{P}=\int\frac{d^3\vec{k}}{(2\pi)^3\,4\omega_k}\vec{k}\left[2\alpha_{\vec{k}}^\dagger\alpha_{\vec{k}}+(2\pi)^3\delta^3(0)\right]$$

The second term is a little disturbing, since it's infinite. On the other hand, it is multiplied by $\vec{k}$, and therefore the positive and negative values of $\vec{k}$ will cancel out, so this infinity can be ignored (at least, I hope so). This can be made more rigorous by moving to finite volume. In any case, we get our final answer:

$$\vec{P}=\int\frac{d^3\vec{k}}{(2\pi)^3\,2\omega_k}\vec{k}\alpha_{\vec{k}}^\dagger\alpha_{\vec{k}}\,.$$

(c) [5] Consider the state $|\vec{p}\rangle=\alpha_{\vec{p}}^\dagger|0\rangle$. Show that it is an eigenstates of $\vec{P}$ and determine its eigenvalue.

We simply let it act on the state and use commutators to simplify:

$$\vec{P}|\vec{p}\rangle=\int\frac{d^3\vec{k}}{(2\pi)^3\,2\omega_k}\vec{k}\alpha_{\vec{k}}^\dagger\alpha_{\vec{k}}\alpha_{\vec{p}}^\dagger|0\rangle=\int\frac{d^3\vec{k}}{(2\pi)^3\,2\omega_k}\vec{k}\alpha_{\vec{k}}^\dagger\left\{\left[\alpha_{\vec{k}},\alpha_{\vec{p}}^\dagger\right]+\alpha_{\vec{p}}^\dagger\alpha_{\vec{k}}\right\}|0\rangle$$

$$=\int\frac{d^3\vec{k}}{(2\pi)^3\,2\omega_k}\vec{k}\alpha_{\vec{k}}^\dagger\left\{(2\pi)^2\,2\omega_k\delta^3\left(\vec{k}-\vec{p}\right)|0\rangle\right\}=\vec{p}\alpha_{\vec{p}}^\dagger|0\rangle=\vec{p}|\vec{p}\rangle$$

So the eigenvalue is $\vec{p}$.

2. [15] A single scalar field has the usual Lagrangian $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$. It is in the state

$$|\psi\rangle = N\exp\left(z\alpha_{\vec{p}}^\dagger\right)|0\rangle = N\sum_{n=0}^{\infty}\frac{z^n\left(\alpha_{\vec{p}}^\dagger\right)^n}{n!}|0\rangle,$$

where z is an arbitrary complex number, and N is a normalization constant.

(a) [4] Show that $\left[\alpha_{\vec{k}},\left(\alpha_{\vec{p}}^\dagger\right)^n\right] = n\left(\alpha_{\vec{p}}^\dagger\right)^{n-1}\left(2\omega_{\vec{p}}\right)(2\pi)^3\delta^3\left(\vec{k}-\vec{p}\right)$.

Formally, this is done by the mathematical process of induction. For $n = 0$, it is trivial to see that both sides vanish. Assume now it is true for n, let's see if we can prove it for $n + 1$:

$$\begin{aligned}\left[\alpha_{\vec{k}},\left(\alpha_{\vec{p}}^\dagger\right)^{n+1}\right] &= \left[\alpha_{\vec{k}},\left(\alpha_{\vec{p}}^\dagger\right)^n\alpha_{\vec{p}}^\dagger\right] = \left(\alpha_{\vec{p}}^\dagger\right)^n\left[\alpha_{\vec{k}},\alpha_{\vec{p}}^\dagger\right] = \left[\alpha_{\vec{k}},\left(\alpha_{\vec{p}}^\dagger\right)^n\right]\alpha_{\vec{p}}^\dagger \\ &= \left(\alpha_{\vec{p}}^\dagger\right)^n\left(2\omega_{\vec{k}}\right)(2\pi)^3\delta^3\left(\vec{k}-\vec{p}\right) + n\left(\alpha_{\vec{p}}^\dagger\right)^{n-1}\left(2\omega_{\vec{p}}\right)(2\pi)^3\delta^3\left(\vec{k}-\vec{p}\right)\alpha_{\vec{p}}^\dagger \\ &= (n+1)\left(\alpha_{\vec{p}}^\dagger\right)^n\left(2\omega_{\vec{p}}\right)(2\pi)^3\delta^3\left(\vec{k}-\vec{p}\right)\end{aligned}$$

and since the formula is appropriate for $n + 1$, we have the inductive step and it is therefore true.

(b) [6] Show that $|\psi\rangle$ is an eigenstates of $\alpha_{\vec{k}}$ and determine its eigenvalue (it will be zero unless $\vec{k} = \vec{p}$).

We have:

$$\begin{aligned}\alpha_{\vec{k}}|\psi\rangle &= N\sum_{n=0}^{\infty}\frac{z^n}{n!}\alpha_{\vec{k}}\left(\alpha_{\vec{p}}^\dagger\right)^n|0\rangle = N\sum_{n=0}^{\infty}\frac{z^n}{n!}\left\{\left[\alpha_{\vec{k}},\left(\alpha_{\vec{p}}^\dagger\right)^n\right] + \left(\alpha_{\vec{p}}^\dagger\right)^n\alpha_{\vec{k}}\right\}|0\rangle \\ &= N\sum_{n=0}^{\infty}\frac{z^n}{n!}n\left(\alpha_{\vec{p}}^\dagger\right)^{n-1}\left(2\omega_{\vec{p}}\right)(2\pi)^3\delta^3\left(\vec{k}-\vec{p}\right)|0\rangle \\ &= N\left(2\omega_{\vec{k}}\right)(2\pi)^3\delta^3\left(\vec{k}-\vec{p}\right)\sum_{n=1}^{\infty}\frac{zz^{n-1}}{(n-1)!}\left(\alpha_{\vec{p}}^\dagger\right)^{n-1}|0\rangle \\ &= z\left(2\omega_{\vec{k}}\right)(2\pi)^3\delta^3\left(\vec{k}-\vec{p}\right)|\psi\rangle\end{aligned}$$

The eigenvalue is infinite when $\vec{k} = \vec{p}$, and zero otherwise, but this doesn't really create a problem. We will also need the Hermitian conjugate of this relationship, which says

$$\langle\psi|\alpha_{\vec{k}}^\dagger = \langle\psi|z^*\left(2\omega_{\vec{k}}\right)(2\pi)^3\delta^3\left(\vec{k}-\vec{p}\right)$$

(c) [5] Assume *N* is chosen so that $|\psi\rangle$ is normalized, that is, $\langle\psi|\psi\rangle = 1$. Evaluate $\langle\psi|\phi(\mathbf{x})|\psi\rangle$.

We substitute it in, and then let $\alpha_{\vec{k}}$ and $\alpha_{\vec{k}}^\dagger$ act to the left and right respectively to simplify. We have

$$\begin{aligned}
\langle\psi|\phi(\mathbf{x})|\psi\rangle &= \langle\psi|\int \frac{d^3\vec{k}}{(2\pi)^3\, 2\omega_k}\left(\alpha_{\vec{k}} e^{i\vec{k}\cdot\vec{x}-i\omega_k t} + \alpha_{\vec{k}}^\dagger e^{-i\vec{k}\cdot\vec{x}+i\omega_k t}\right)|\psi\rangle \\
&= \langle\psi|\int \frac{d^3\vec{k}}{(2\pi)^3\, 2\omega_k}\left(2\omega_{\vec{k}}\right)(2\pi)^3\,\delta^3\left(\vec{k}-\vec{p}\right)\left(z e^{i\vec{k}\cdot\vec{x}-i\omega_k t} + z^* e^{-i\vec{k}\cdot\vec{x}+i\omega_k t}\right)|\psi\rangle \\
&= \left(z e^{-i\mathbf{p}\cdot\mathbf{x}} + z^* e^{i\mathbf{p}\cdot\mathbf{x}}\right)\langle\psi|\psi\rangle = 2\,\mathrm{Re}\left(z e^{-i\mathbf{p}\cdot\mathbf{x}}\right).
\end{aligned}$$

We can't really simplify this more, since *z* might be complex.