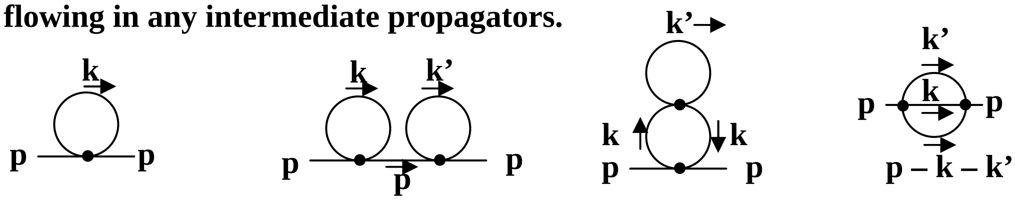
Physics 744 - Field Theory

Solution Set 6

1. [15] In the $\lambda\phi^4$ theory:

(a) [3] Draw all connected Feynman diagrams at the two loop level with a single ϕ particle with momentum p entering on the left and leaving on the right. There should be one diagram at the one-loop level, and three at the two loop level. Do *not* take advantage of normal ordering. Include the appropriate symmetry factor.

(b) [4] Label all intermediate momenta. Indicate which way the momentum is flowing in any intermediate propagators.



The four diagrams are sketched above. The first has a symmetry factor of $\frac{1}{2}$, as we worked out in class, because the loop can be turned around. The second one has a factor of $\frac{1}{4}$, since it has two such loops, while the third one also has a factor of $\frac{1}{4}$, because you can twist around the top loop and you can also interchange the two legs on the bottom loop. The final diagram has a symmetry factor of $\frac{1}{6}$, because the three intermediate legs are all interchangeable.

I have included the momentum labels on each loop. Which way you draw the arrows is arbitrary. In the first two diagrams, there is one arbitrary momentum, and for the second one there are two. In the third diagram there are again two, and it is clear that both sides of the lower loop have the same momentum **k** flowing around it. In the final diagram, you can pick two of the intermediate legs to have arbitrary momenta, and the third leg can then be determined by conservation of momentm.

(c) [8] Write the Feynman amplitude in each case. Do *not* attempt to perform the integrals.

The Feynman amplitude is, in this case,

$$i\mathcal{M} = \frac{1}{2}\int \frac{d^4\mathbf{k}}{(2\pi)^4}\frac{(-i\lambda)i}{\mathbf{k}^2 - m^2 + i\varepsilon} + \frac{1}{4}\int \frac{d^4\mathbf{k}}{(2\pi)^4}\int \frac{d^4\mathbf{k}'}{(2\pi)^4}\frac{(-i\lambda)i}{\mathbf{k}^2 - m^2 + i\varepsilon}\frac{(-i\lambda)i}{\mathbf{k}'^2 - m^2 + i\varepsilon}\frac{i}{\mathbf{p}^2 - m^2 + i\varepsilon}$$

$$+\frac{1}{4}\int \frac{d^4\mathbf{k}}{(2\pi)^4}\int \frac{d^4\mathbf{k}'}{(2\pi)^4}\frac{(-i\lambda)i^2}{\left(\mathbf{k}^2 - m^2 + i\varepsilon\right)^2}\frac{(-i\lambda)i}{\mathbf{k}'^2 - m^2 + i\varepsilon}$$

$$+\frac{1}{6}\int \frac{d^4\mathbf{k}}{(2\pi)^4}\int \frac{d^4\mathbf{k}'}{(2\pi)^4}\frac{(-i\lambda)^2 i^3}{\left(\mathbf{k}^2 - m^2 + i\varepsilon\right)\left(\mathbf{k}'^2 - m^2 + i\varepsilon\right)\left[\left(\mathbf{p} - \mathbf{k} - \mathbf{k}'\right)^2 - m^2 + i\varepsilon\right]}$$

with each term corresponding to each of the four diagrams.

2. [10] Suppose you have two particles in the final state for some diagram. You need to know the momentum of the final state particles. Assume the total center of mass energy is *E*.

(a) [6] Show that the momenta of the final state particles can be written as

$$p = \frac{1}{2E}\sqrt{E^4 + m_1^4 + m_2^4 - 2E^2 m_1^2 - 2E^2 m_2^2 - 2m_1^2 m_2^2}$$

Well, the two momenta are equal and opposite, so call this p. The energies of the two particles are $E_1 = \sqrt{p^2 + m_1^2}$ and $E_2 = \sqrt{p^2 + m_2^2}$. We therefore have

$$E = \sqrt{p^2 + m_1^2} + \sqrt{p^2 + m_2^2},$$

$$\left(E - \sqrt{p^2 + m_1^2}\right)^2 = p^2 + m_2^2,$$

$$E^2 - 2E\sqrt{p^2 + m_1^2} + p^2 + m_1^2 = p^2 + m_2^2,$$

$$2E\sqrt{p^2 + m_1^2} = m_2^2 - m_1^2 - E^2,$$

$$4E^2 p^2 + 4E^2 m_1^2 = m_2^4 + m_1^4 + E^4 - 2m_1^2 m_2^2 - 2m_2^2 E^2 + 2m_1^2 E^2,$$

$$2Ep = \sqrt{m_2^4 + m_1^4 + E^4 - 2m_1^2 m_2^2 - 2m_2^2 E^2 - 2m_1^2 E^2},$$

$$p = \frac{1}{2E}\sqrt{m_2^4 + m_1^4 + E^4 - 2m_1^2 m_2^2 - 2m_2^2 E^2 - 2m_1^2 E^2}.$$

Obviously, I messed up the formula, but you can easily see what I meant.

(b) [4] Simplify this formula in the cases (i) $m_2 = 0$ and (ii) $m_1 = m_2 = m$.

If $m_2 = 0$, we have

$$p = \frac{1}{2E}\sqrt{E^4 + m_1^4 - 2E^2 m_1^2} = \frac{1}{2E}\sqrt{\left(E^2 - m_1^2\right)^2} = \frac{E^2 - m_1^2}{2E}$$

If $m_1 = m_2 = m$, we have

$$p = \frac{1}{2E}\sqrt{E^4 + 2m^4 - 4E^2 m^2 - 2m^4} = \frac{E}{2E}\sqrt{E^2 - 4m^2} = \sqrt{\tfrac{1}{4}E^2 - m^2}.$$

3. [25] The Feynman invariant amplitude for muon decay:

$$\mu^-(\mathbf{p}_\mu) \to e^-(\mathbf{p}_1) + \bar{\nu}_e(\mathbf{p}_2) + \nu_\mu(\mathbf{p}_3)$$

is given by

$$|i\mathcal{M}|^2 = 64G_F^2(\mathbf{p}_\mu \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_3).$$

Compared to the muon, the other particles are so light they can be treated as massless. The constant $G_F = 1.17\times 10^{-5}\ \text{GeV}^{-2}$ **is a constant involved in weak decays called *Fermi's constant*.**

(a) [6] Use conservation of four-momentum to relate the two dot-products appearing in this formula. Working in the rest frame of the muon, write out this matrix element explicitly and show that it depends on only *one* of the final state energies.

By conservation of four-momentum, $\mathbf{p}_\mu = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3$. Rearranging, this implies $\mathbf{p}_\mu - \mathbf{p}_2 = \mathbf{p}_1 + \mathbf{p}_3$, which we then square to yield

$$(\mathbf{p}_\mu - \mathbf{p}_2)^2 = (\mathbf{p}_1 + \mathbf{p}_3)^2,$$

$$\mathbf{p}_\mu^2 + \mathbf{p}_2^2 - 2\mathbf{p}_\mu\cdot\mathbf{p}_2 = \mathbf{p}_1^2 + \mathbf{p}_3^2 + 2\mathbf{p}_1\cdot\mathbf{p}_3,$$

$$m_\mu^2 = 2\mathbf{p}_\mu\cdot\mathbf{p}_2 + 2\mathbf{p}_1\cdot\mathbf{p}_3.$$

Now, in the rest frame of the muon, $\mathbf{p}_\mu = (m_\mu, 0, 0, 0)$, so that $\mathbf{p}_\mu\cdot\mathbf{p}_2 = m_\mu E_2$. It follows that

$$\mathbf{p}_1\cdot\mathbf{p}_3 = \tfrac{1}{2}m_\mu^2 - \mathbf{p}_\mu\cdot\mathbf{p}_2 = \tfrac{1}{2}m_\mu^2 - m_\mu E_2.$$

Substituting into the Feynman amplitude, we find

$$|i\mathcal{M}|^2 = 64G_F^2(\mathbf{p}_\mu\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_3) = 64G_F^2 m_\mu E_2\left(\tfrac{1}{2}m_\mu^2 - m_\mu E_2\right) = 32G_F^2 m_\mu^2(m_\mu - 2E_2)E_2$$

(b) [6] Since the final state particles are massless, the three-momenta have magnitudes equal to their energies. Use this plus conservation of energy to show that none of the final state particles has an energy greater than $\tfrac{1}{2}m_\mu$**. Use this to write *three* inequalities on the energies E_1 and E_2. Sketch the allowed region in E_1-E_2 space.**

Well, the three momenta are vectors whose sum must add up to zero; *i.e.*, they form a triangle. The *sum* of the magnitudes must be the total energy, which is m_μ. So we have

$$\vec{p}_1 + \vec{p}_2 + \vec{p}_3 = 0, \quad \text{and} \quad |\vec{p}_1| + |\vec{p}_2| + |\vec{p}_3| = m_\mu$$

The triangle inequality tells you that the sum of two sides is always longer than the third side. It follows that $\left|\vec{p}_1\right|+\left|\vec{p}_2\right|\ge\left|\vec{p}_3\right|$, and therefore

$$\left|\vec{p}_1\right|+\left|\vec{p}_2\right|+\left|\vec{p}_3\right|\ge 2\left|\vec{p}_3\right|=2E_3,$$
$$m_\mu \ge 2E_3 .$$

Of course, the same thing is true of the other two energies, so we have

$$E_1, E_2, E_3 \le \tfrac{1}{2}m_\mu .$$

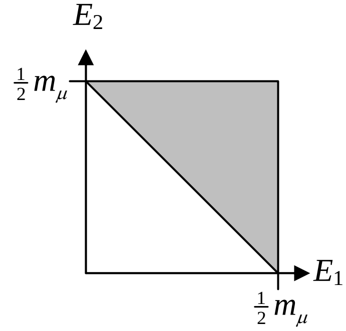
This looks like only two restrictions on the energies E_1 and E_2, but recall by conservation of energy that $E_3 = m_\mu - E_1 - E_2$. Hence we have

$$m_\mu - E_1 - E_2 \le \tfrac{1}{2}m_\mu ,$$
$$E_1 + E_2 \ge \tfrac{1}{2}m_\mu .$$

Our three inequalities are therefore

$$E_1 \le \tfrac{1}{2}m_\mu ,\quad E_2 \le \tfrac{1}{2}m_\mu ,\quad \text{and}\quad E_1 + E_2 \ge \tfrac{1}{2}m_\mu .$$

The corresponding allowed region is the shaded region in the sketch at right.



(c) [8] Do the final state integrals and determine the decay rate of the muon. The lifetime is the reciprocal of this.

We must calculate the final state integrals. The tricky part is figuring out the limits of integration. For fixed E_2, the range for E_1 is clearly from $\tfrac{1}{2}m_\mu - E_2$ to $\tfrac{1}{2}m_\mu$. The unrestricted range for E_2 is then from 0 to $\tfrac{1}{2}m_\mu$. We therefore have

$$\Gamma = \frac{D}{2m_\mu} = \frac{1}{2m_\mu}\frac{1}{8(2\pi)^5}\int dE_1\, dE_2\, d\phi_1\, d\cos\theta_1\, d\phi_{12}\left|i\mathcal{M}\right|^2$$
$$= \frac{1}{2m_\mu}\frac{2(2\pi)^2}{8(2\pi)^5}\int_0^{\frac{1}{2}m_\mu} dE_2 \int_{\frac{1}{2}m_\mu - E_2}^{\frac{1}{2}m_\mu} dE_1\left[32G_F^2 m_\mu^2\left(m_\mu - 2E_2\right)E_2\right]$$
$$= \frac{32G_F^2 m_\mu^2}{8m_\mu(2\pi)^3}\int_0^{\frac{1}{2}m_\mu}\left(m_\mu - 2E_2\right)E_2^2\, dE_2 = \frac{G_F^2 m_\mu}{2\pi^3}\left(\tfrac{1}{3}m_\mu E_2^3 - \tfrac{1}{2}E_2^4\right)\Big|_0^{\frac{1}{2}m_\mu}$$
$$= \frac{1}{2\pi^3}G_F^2 m_\mu^5\left(\tfrac{1}{24}-\tfrac{1}{32}\right) = \frac{G_F^2 m_\mu^5}{192\pi^3}$$

(d) [5] The mass of the muon is $m_\mu = 0.1057\ \text{GeV}$. Find the lifetime $\tau = \Gamma^{-1}$ in seconds and compare to the experimental lifetime of $\tau = 2.197\ \mu\text{s}$.

Substituting in the explicit values and throwing in a factor of $\hbar = 6.528\times 10^{-25}\ \text{GeV}\cdot\text{s}$, we have

$$\Gamma = \frac{\left(1.17\times 10^{-5}\ \text{GeV}^{-2}\right)^2 \left(0.1057\ \text{GeV}\right)^5}{192\pi^3\left(6.528\times 10^{-25}\ \text{GeV}\cdot\text{s}\right)} = 4.64\times 10^5\ \text{s}^{-1}$$

Taking the reciprocal, we have $\tau = \Gamma^{-1} = 2.15\,\mu\text{s}$. That's pretty close, really! Not sure why there's a remaining discrepancy, though part of the problem was that I didn't give you G_F accurately enough.