Quantum Mechanics 741 – Midterm Equations

The following equations you should memorize, and understand how to use them:

Basic Equations:

$$E = \hbar\omega = hf$$
$$\mathbf{p} = \hbar\mathbf{k}$$

Important Operators

$$\mathbf{R}\psi = \mathbf{r}\psi$$
$$\mathbf{P}\psi = -i\hbar\nabla\psi$$
$$\mathbf{L} = \mathbf{R}\times\mathbf{P}$$

Schrödinger

$$i\hbar\frac{\partial}{\partial t}\left|\Psi(t)\right\rangle = H\left|\Psi(t)\right\rangle$$
$$E_n\left|\psi_n\right\rangle = H\left|\psi_n\right\rangle$$
$$\left|\Psi(t)\right\rangle = \sum_n c_n e^{-iE_n t/\hbar}\left|\phi_n\right\rangle$$
$$c_n = \left\langle\phi_n\middle|\Psi(0)\right\rangle$$

Hamiltonian:

$$H = \mathbf{P}^2/(2m) + V$$

Commutators

$$[A,B] = AB - BA$$
$$[R_i, P_j] = i\hbar\delta_{ij}$$
$$[R_i, R_j] = 0 = [P_i, P_j]$$

Infinite Square Well

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi n x}{a}\right)$$
$$E_n = \frac{\pi^2\hbar^2 n^2}{2ma^2}$$

Complete Ortho-normal Basis

$$\left\langle\phi_i\middle|\phi_j\right\rangle = \delta_{ij}$$
$$\sum_i\left|\phi_i\right\rangle\left\langle\phi_i\right| = 1$$
$$\left\langle\beta\middle|\alpha\right\rangle = \delta(\beta-\alpha)$$
$$\int d\alpha\left|\alpha\right\rangle\left\langle\alpha\right| = 1$$

Probability Density

$$\rho(\mathbf{r}) = \left|\psi(\mathbf{r})\right|^2$$
$$P(a<x<b) = \int_a^b\left|\psi(x)\right|^2 dx$$
$$1 = \int\left|\psi(\mathbf{r})\right|^2 d^3\mathbf{r}$$

Expectation values and uncertainties

$$\bar{A} = \langle A\rangle = \langle\Psi|A|\Psi\rangle$$
$$(\Delta A)^2 = \langle A^2\rangle - \langle A\rangle^2$$
$$\Delta A\,\Delta B \ge \tfrac{1}{2}\left|\left\langle i[A,B]\right\rangle\right|$$
$$\Delta X\,\Delta P \ge \tfrac{1}{2}\hbar$$

Harm. Oscillator

$$V = \tfrac{1}{2}m\omega^2X^2$$
$$|n\rangle : n = 0,1,2,\dots$$
$$E_n = \hbar\omega\left(n+\tfrac{1}{2}\right)$$
$$\left[a, a^\dagger\right] = 1$$

Symmetry

$$T(\mathbf{a})|\mathbf{r}\rangle = |\mathbf{r}+\mathbf{a}\rangle$$
$$R(\mathcal{R})|\mathbf{r}\rangle = |\mathcal{R}\mathbf{r}\rangle$$
$$\Pi|\mathbf{r}\rangle = |-\mathbf{r}\rangle$$
$$T^\dagger(\mathbf{a})T(\mathbf{a}) = 1$$
$$R^\dagger(\mathcal{R})R(\mathcal{R}) = 1$$

Conjugation

$$|\psi\rangle^\dagger = \langle\psi|, \quad \langle\psi|^\dagger = |\psi\rangle$$
$$\langle\psi|A^\dagger|\phi\rangle = \langle\phi|A|\psi\rangle^*$$
$$A^\dagger = A^{*\mathrm{T}}, \quad c^\dagger = c^*$$

Translation

$$[P_x, H] = 0$$
$$P_x|k_x\rangle = \hbar k_x|k_x\rangle$$
$$\psi = \phi(y,z)e^{ik_x x}$$

Rotation (2D)

$$[L_z, H] = 0$$
$$L_z|\psi\rangle = \hbar m|\psi\rangle$$
$$m = 0, \pm1, \pm2, \dots$$
$$\psi(\rho,\theta) = R(\rho)e^{im\theta}$$

Some Bras

$$\langle\mathbf{r}|\psi\rangle = \psi(\mathbf{r})$$
$$\langle\phi|\psi\rangle \equiv \int\phi^*(\mathbf{r})\psi(\mathbf{r})\,d^3\mathbf{r}$$

Measure A and get a:

$$P(a) = \sum_n\left|\langle a,n|\Psi\rangle\right|^2$$
$$\left|\Psi^+\right\rangle = \frac{\sum_n|a,n\rangle\langle a,n|\Psi\rangle}{\sqrt{P(a)}}$$

Other things you should know:

- How to do integrals in Cartesian and Spherical coordinates
- The general form of a wave function if its second derivative is proportional to itself or proportional to minus itself
- Which solutions are physically meaningful in typical situations (not divergent at infinity; incoming, reflected, and transmitted waves)
- How to match boundary conditions across a boundary, with or without a delta-function in the potential
- How to generalize from 1D potentials to many-D potentials (infinite square well and harmonic oscillator)
- How to deal with Hermitian conjugation of arbitrary expressions
- How to work out complicated commutators from simple ones

- The ideas behind coordinate bases, especially, orthonormal bases
- Matrix notation
- Hermitian operators have real eigenvalues, unitary have ones of magnitude one
- Both Hermitian operators and unitary operators have orthogonal eigenvectors
- How to find the eigenvalues and normalized eigenvectors of a matrix
- How to estimate ground state energies using the uncertainty principle
- If you perform a measurement of the operator A and get the result a, you are automatically in an eigenstate with this eigenvalue. If there is only one eigenvector with this eigenvalue, that's the state you are in.
- How to deal with multiple harmonic oscillators, coupled or uncoupled
- How to tell from the potential if there is a translation or a rotation symmetry (continuous or discrete)
- Symmetry operations have eigenvalues of magnitude one
- If repeating a symmetry operation N times brings it back where it started, then the eigenvalue must be an N'th root of one
- Complex eigenvalues for symmetry operations imply degenerate states

The following equations you need not memorize, but you should know how to use them if given to you:

Reflection from a step:

$$R = \left(\frac{\sqrt{E} - \sqrt{E - V_0}}{\sqrt{E} + \sqrt{E - V_0}} \right)^2$$

$$T = 1 - R$$

Probability Current:

$$\mathbf{j} \equiv -\frac{i\hbar}{2m}\left(\Psi^* \nabla \Psi - \Psi \nabla \Psi^* \right)$$

$$\mathbf{j} = \frac{\hbar}{m} \operatorname{Im}\left(\Psi^* \nabla \Psi \right)$$

Momentum bra

$$\langle \mathbf{k} | \psi \rangle = \int \frac{d^3\mathbf{r}}{(2\pi)^{3/2}} e^{-i\mathbf{k}\cdot\mathbf{r}} \psi(\mathbf{r})$$

Commutators

$$[AB, C] = A[B, C] + [A, C]B$$

$$[A, BC] = B[A, C] + [A, B]C$$

Evolution of Expectation

$$\frac{d}{dt}\langle A \rangle = \frac{i}{\hbar}\langle [H, A] \rangle + \left\langle \frac{\partial A}{\partial t} \right\rangle$$

$$\frac{d}{dt}\langle \mathbf{R} \rangle = \frac{1}{m}\langle \mathbf{P} \rangle$$

$$\frac{d}{dt}\langle \mathbf{P} \rangle = \langle -\nabla V(\mathbf{R}) \rangle$$

Rotation Matrices 2D

$$\mathcal{R}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Harmonic Oscillator

$$X = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger \right)$$

$$P = i\sqrt{\frac{\hbar m\omega}{2}}\left(a^\dagger - a \right)$$

$$a|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$a^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle$$

Symmetry Operations

$$T(\mathbf{a}) = \exp(-i\,\mathbf{a}\cdot\mathbf{P}/\hbar)$$

$$R(\mathcal{R}(\hat{\mathbf{n}}, \theta)) = \exp(-i\theta\,\hat{\mathbf{n}}\cdot\mathbf{L}/\hbar)$$

$$T^\dagger(\mathbf{a}) V(\mathbf{R}) T(\mathbf{a}) = V(\mathbf{R} + \mathbf{a})$$

$$R^\dagger(\mathcal{R}) V(\mathbf{R}) R^\dagger(\mathcal{R}) = V(\mathcal{R}\mathbf{R})$$

$$T^\dagger(\mathbf{a}) \mathbf{P} T(\mathbf{a}) = \mathbf{P}$$

$$R^\dagger(\mathcal{R}) \mathbf{P} R(\mathcal{R}) = \mathcal{R}\mathbf{P}$$

$$R^\dagger(\mathcal{R}) \mathbf{P}^2 R(\mathcal{R}) = \mathbf{P}^2$$

Coherent States:

$$a|z\rangle = z|z\rangle, \quad \langle z| a^\dagger = \langle z| z^*$$

N'th roots of unity:

$$e^{2\pi i j/N},\ j = 0, 1, \ldots, N-1$$