Quantum Mechanics 742 – Second Test
Covering Chapters 14 – 18

The following new equations you should memorize, and understand how to use them:

Cross-section
$\Gamma = n\sigma|\Delta \mathbf{v}|$
$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega$

Momentum Change
$\mathbf{K} = k\hat{\mathbf{r}} - k\hat{\mathbf{z}}$

Fermi's Golden Rule: $\mathcal{T}_{FI} = W_{FI} + \cdots$
$\Gamma(I \to F) = 2\pi\hbar^{-1} |\mathcal{T}_{FI}|^2 \delta(E_F - E_I)$

Time Dependent Perturbation Theory
$H = H_0 + W(t)$
$H_0|\phi_n\rangle = E_n|\phi_n\rangle$
$\omega_{nm} \equiv (E_n - E_m)/\hbar$
$W_{nm}(t) \equiv \langle\phi_n|W(t)|\phi_m\rangle$
$P(I \to F) = |S_{FI}|^2$

Sudden: $T\Delta E \ll \hbar$
$P(I \to F) = |\langle\psi_F|\psi_I\rangle|^2$
Adiabatic: $T\Delta E \gg \hbar$
$P(I \to F) = \delta_{IF}$

Coulomb Gauge
$\nabla \cdot \mathbf{A} = 0$

Photon Creation and Annihilation Operators
$\left[a_{\mathbf{k}\sigma}, a_{\mathbf{k}'\sigma'}^\dagger\right] = \delta_{\mathbf{k},\mathbf{k}'}\delta_{\sigma,\sigma'}$
$a_{\mathbf{k}\sigma}|n,\mathbf{k},\sigma\rangle = \sqrt{n}|n-1,\mathbf{k},\sigma\rangle$
$a_{\mathbf{k}\sigma}^\dagger|n,\mathbf{k},\sigma\rangle = \sqrt{n+1}|n+1,\mathbf{k},\sigma\rangle$

Electric Dipole Moment
$\mathbf{r}_{nm} \equiv \langle\phi_n|\mathbf{R}|\phi_m\rangle$

EM waves
$\boldsymbol{\varepsilon}_{\mathbf{k}\sigma} \cdot \boldsymbol{\varepsilon}_{\mathbf{k}\sigma'}^* = \delta_{\sigma\sigma'}$
$\boldsymbol{\varepsilon}_{\mathbf{k}\sigma} \cdot \mathbf{k} = 0$
$\omega_k = ck$

EM Hamiltonian
$H = \sum_{\mathbf{k},\sigma} \hbar\omega_k a_{\mathbf{k}\sigma}^\dagger a_{\mathbf{k}\sigma}$

Converting Finite → Infinite
$\lim_{V\to\infty}\left[\frac{1}{V}\sum_{\mathbf{k}} f(\mathbf{k})\right] = \int f(\mathbf{k}) \frac{d^3\mathbf{k}}{(2\pi)^3}$

Other things you should know:

- The meaning of cross-section and differential cross-section
- Doing integrals in spherical coordinates
- How to do integrals like $\int f(x)\delta[g(x)]dx$
- In the adiabatic approximation, make sure the eigenstates you use are correct eigenstates of the initial and final Hamiltonian.
- In the adiabatic approximation: The lowest energy state goes to the lowest state, second lowest to second lowest, etc.
    - However, if there is a symmetry that always commutes with the Hamiltonian, then first sort states by eigenvalues of that symmetry
- For harmonic perturbations, make sure you know how to extract W given $W(t)$, and don't get the two confused
    - In contrast, when the perturbation is independent of time, $W(t) = W$
- How to compute expressions like $\boldsymbol{\varepsilon}_{\mathbf{k}\sigma} \cdot \mathbf{r}_{FI}$ for each of the two possible polarizations
- How to average (or sum) over polarizations, and average (or integrate) over angles
- How to take the limit $V \to \infty$, and turn sums over final states into integrals you can do
- What a quantum state like $|n_1,\mathbf{k}_1,\sigma_1;n_2,\mathbf{k}_2,\sigma_2\rangle$ means, or $|\phi_a;n_1,\mathbf{k}_1,\sigma_1;n_2,\mathbf{k}_2,\sigma_2\rangle$ means
- Understanding that after quantizing the EM field, electric and magnetic fields are now operators that are functions of $\mathbf{r}$, but not t
- How to write out sums with EM fields and expectation values where the sums collapse to few terms
- How to find the energy of a system of photons, or photons plus an atom
    - Related: the time dependence of such a system
- How to create or annihilate one photon from any state with any number of photons
- Qualitatively, what our diagrams mean in our computations

- Why we concentrate only on certain diagrams when scattering near a resonance

The following new equations you need not memorize, but you should know how to use them if given to you:

Time-dependent Perturbation Theory

$$S_{FI} = \delta_{FI} + (i\hbar)^{-1}\int_0^T dt\, W_{FI}(t) e^{i\omega_{FI}t} + (i\hbar)^{-2}\sum_n \int_0^T dt\, W_{Fn}(t) e^{i\omega_{Fn}t}\int_0^t dt'\, W_{nI}(t') e^{i\omega_{nI}t'} + \cdots$$

Electric Dipole Absorption

$$\Gamma_{E1} = 4\pi^2\alpha\hbar^{-1}\mathcal{I}(\omega_{FI})\left|\boldsymbol{\varepsilon}\cdot\mathbf{r}_{FI}\right|^2$$

Harmonic Perturbations:

$$W(t) = We^{-i\omega t} + W^\dagger e^{i\omega t}, \quad W_{FI} \equiv \langle\phi_F|W|\phi_I\rangle$$

$$\Gamma(I\to F) = \begin{cases} 2\pi\hbar^{-1}\left|W_{FI}\right|^2\delta(E_F - E_I - \hbar\omega) & \text{if } E_F > E_I \\ 2\pi\hbar^{-1}\left|W_{FI}^\dagger\right|^2\delta(E_F - E_I + \hbar\omega) & \text{if } E_F < E_I \end{cases}$$

Scattering Away from Resonance

$$\mathcal{T}_{FI} = \frac{-e^2}{\varepsilon_0 V}\sum_n \frac{\omega\omega_{nI}}{\omega_{nI}^2 - \omega^2}\left(\boldsymbol{\varepsilon}_F^*\cdot\mathbf{r}_{nI}^*\right)\left(\boldsymbol{\varepsilon}_I\cdot\mathbf{r}_{nI}\right)$$

Transition matrix:

$$\mathcal{T}_{FI} = W_{FI} + \lim_{\varepsilon\to 0^+}\left[\sum_n \frac{W_{Fn}W_{nI}}{(E_I - E_n + i\varepsilon)} + \sum_n\sum_m \frac{W_{Fn}W_{nm}W_{mI}}{(E_I - E_m + i\varepsilon)(E_I - E_n + i\varepsilon)} + \cdots\right]$$

Spontaneous Decay

$$\Gamma = \frac{4\alpha}{3c^2}\omega_{IF}^3\left|\mathbf{r}_{FI}\right|^2$$

Dirac Equation

$$i\hbar\frac{\partial}{\partial t}\Psi = c\boldsymbol{\alpha}\cdot(\mathbf{P} + e\mathbf{A})\Psi - eU\Psi + mc^2\beta\Psi$$

$$\beta = \begin{pmatrix} \mathbf{0} & -\mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}, \quad \boldsymbol{\alpha} = \begin{pmatrix} \boldsymbol{\sigma} & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{\sigma} \end{pmatrix}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Electromagnetic Operators

$$\mathbf{A}(\mathbf{r}) = \sum_{\mathbf{k},\sigma}\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_k}}\left(a_{\mathbf{k}\sigma}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}} + a_{\mathbf{k}\sigma}^\dagger\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}^* e^{-i\mathbf{k}\cdot\mathbf{r}}\right)$$

$$\mathbf{E}(\mathbf{r}) = \sum_{\mathbf{k},\sigma}\sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\,i\left(a_{\mathbf{k}\sigma}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}} - a_{\mathbf{k}\sigma}^\dagger\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}^* e^{-i\mathbf{k}\cdot\mathbf{r}}\right)$$

$$\mathbf{B}(\mathbf{r}) = \sum_{\mathbf{k},\sigma}\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_k}}\,i\mathbf{k}\times\left(a_{\mathbf{k}\sigma}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}} - a_{\mathbf{k}\sigma}^\dagger\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}^* e^{-i\mathbf{k}\cdot\mathbf{r}}\right)$$

Lab vs. CM frame

$$\left(\frac{d\sigma}{d\Omega}\right)_L = \left(\frac{d\sigma}{d\Omega}\right)\frac{\left(1 + 2\gamma\cos\theta + \gamma^2\right)^{3/2}}{1 + \gamma\cos\theta}$$

$$\gamma = \frac{m}{M}$$

Scattering Formulas

$$k^2 = \frac{2\mu E}{\hbar^2}$$

Scattering Near Resonance

$$\sigma = \frac{8\pi\alpha^2\omega_{nI}^4\left|\mathbf{r}_{nI}\right|^4}{3c^2\left[(\omega - \omega_{nI})^2 + \frac{1}{4}\Gamma^2\right]}$$

1st Born Approximation

$$\frac{d\sigma}{d\Omega} = \frac{\mu^2}{4\pi^2\hbar^4}\left|\int d^3\mathbf{r}\, V(\mathbf{r})e^{-i\mathbf{K}\cdot\mathbf{r}}\right|^2$$