Quantum Mechanics
Graduate Exam

Summer, 2022

Each problem is worth 25 points. The points for individual parts are marked in square brackets. **To ensure full credit, show your work.** Do any four (4) of the following five (5) problems. If you attempt all 5 problems you must clearly state which 4 problems you want to have graded.

1. A particle of mass m is in the potential $V(x)=\begin{cases}\lambda\delta\left(x-\frac{1}{2}a\right) & 0<x<a,\\ \infty & \text{otherwise},\end{cases}$ where λ is small.

   (a) [5] What are the exact eigenstate wave functions and energies of the states if $\lambda=0$?

   (b)[20] What is the ground state eigenstate to first order and its energy to second order in λ? You may leave the eigenstate as an infinite sum, but for full credit you must perform any sums explicitly (possibly helpful formulas below)

Sum Formulas

$$\sum_{n=2}^{\infty}\frac{1}{n^2-1}=\frac{3}{4},\quad \sum_{n=1}^{\infty}\frac{1}{(2n)^2-1}=\frac{1}{2},\quad \sum_{n=1}^{\infty}\frac{1}{(2n+1)^2-1}=\frac{1}{4},$$

$$\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}=\frac{1}{4},\quad \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)^2-1}=\frac{1}{2}-\frac{\pi}{4},\quad \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)^2-1}=\frac{1}{4}-\frac{\ln(2)}{2},$$

2. The electron is in a hydrogen atom is in the 3d orbital in the state $\left|l,s,m_l,m_s\right\rangle=\left|2,\frac{1}{2},1,-\frac{1}{2}\right\rangle$.

   (a) [7] If you measure $\mathbf{J}^2=(\mathbf{L}+\mathbf{S})^2$ in this state, what are the corresponding possible j-values and their corresponding probabilities? You may consult the Clebsch-Gordan coefficient values given below.

   (b) [6] For each outcome in part (a), compute the value of $\mathbf{L}\cdot\mathbf{S}=\frac{1}{2}\left(\mathbf{J}^2-\mathbf{L}^2-\mathbf{S}^2\right)$.

   (c) [6] Combining the above information, deduce the expectation value of $\mathbf{L}\cdot\mathbf{S}$ for the initial state given.

   (d) [6] Finally, show that $\mathbf{L}\cdot\mathbf{S}=\frac{1}{2}\left(L_+S_-+L_-S_+\right)+L_zS_z$, where $L_\pm=L_x\pm iL_y$ and $S_\pm=S_x\pm iS_y$, and use this to check the result by computing $\left\langle 2,\frac{1}{2},1,-\frac{1}{2}\right|\mathbf{L}\cdot\mathbf{S}\left|2,\frac{1}{2},1,-\frac{1}{2}\right\rangle$ in the original basis.

Raising and Lowering Angular Momentum: $J_\pm\left|j,m\right\rangle=\hbar\sqrt{j^2+j-m^2\mp m}\left|j,m\pm1\right\rangle$

Clebsch-Gordan Coefficients: $\left\langle j_1j_2;m_1m_2\middle|jm\right\rangle$ for $j_1,j_2=2,\frac{1}{2}$ and $m>0$: $\left\langle 2\tfrac{1}{2};2\tfrac{1}{2}\middle|\tfrac{5}{2}\tfrac{5}{2}\right\rangle=1$

$$\left\langle 2\tfrac{1}{2};1\tfrac{1}{2}\middle|\tfrac{5}{2}\tfrac{3}{2}\right\rangle=\sqrt{\tfrac{4}{5}},\quad \left\langle 2\tfrac{1}{2};2\tfrac{-1}{2}\middle|\tfrac{5}{2}\tfrac{3}{2}\right\rangle=\sqrt{\tfrac{1}{5}},\quad \left\langle 2\tfrac{1}{2};1\tfrac{1}{2}\middle|\tfrac{3}{2}\tfrac{3}{2}\right\rangle=\sqrt{\tfrac{1}{5}},\quad \left\langle 2\tfrac{1}{2};2\tfrac{-1}{2}\middle|\tfrac{3}{2}\tfrac{3}{2}\right\rangle=-\sqrt{\tfrac{4}{5}},$$

$$\left\langle 2\tfrac{1}{2};0\tfrac{1}{2}\middle|\tfrac{5}{2}\tfrac{1}{2}\right\rangle=\sqrt{\tfrac{3}{5}},\quad \left\langle 2\tfrac{1}{2};1\tfrac{-1}{2}\middle|\tfrac{5}{2}\tfrac{1}{2}\right\rangle=\sqrt{\tfrac{2}{5}},\quad \left\langle 2\tfrac{1}{2};0\tfrac{1}{2}\middle|\tfrac{3}{2}\tfrac{1}{2}\right\rangle=\sqrt{\tfrac{2}{5}},\quad \left\langle 2\tfrac{1}{2};1\tfrac{-1}{2}\middle|\tfrac{3}{2}\tfrac{1}{2}\right\rangle=-\sqrt{\tfrac{3}{5}}.$$

3. Estimate the ground state energy of the hydrogen atom (potential $V = -\beta/r$), by the variational method using the trial wave function $\psi(r) = \begin{cases} a-r & \text{if } r < a, \\ 0 & \text{if } r > a, \end{cases}$ and compare the resulting value with the exact answer $E = -\beta^2 m/(2\hbar^2)$.

<u>Possibly Helpful Formula</u> $\nabla^2\psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$

4. Imagine a system in which there are just two linearly independent states $\{|1\rangle, |2\rangle\}$, and the Hamiltonian, represented as a matrix in this basis, is $\hat{H} = \begin{pmatrix} h & g \\ g & h \end{pmatrix}$, where g and h are real constants.
   (a) [9] Find the eigenvalues and (normalized) eigenvectors of this Hamiltonian.
   (b) [9] Suppose the system starts out at $t = 0$ in state $|1\rangle$. What is the state at time t?
   (c) [7] What is the probability that it will still be in state $|1\rangle$ at time $t = \pi\hbar/2g$?

5. An electron is in the spin state $|\chi\rangle = A\begin{pmatrix} 3i \\ 4 \end{pmatrix}$ in the standard basis of eigenstates of S_z.
   (a) [4] Determine the normalization constant A.
   (b) [8] Find the expectation value of S_x, S_y, and S_z.
   (c) [7] Find the uncertainties for these measurements.
   (d) [6] Confirm your results are consistent with all three generalized uncertainty relations for these observables.

<u>Possibly Helpful Formulas:</u>

$$S_i = \tfrac{1}{2}\hbar\sigma_i,\quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\quad \sigma_i^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$