Solutions to Quantum Mechanics Graduate Exam Summer 2024

Each problem is worth 25 points. The points for individual parts are marked in square brackets. **To ensure full credit, show your work.** Do any four (4) of the following five (5) problems. If you attempt all 5 problems, you must clearly state which 4 problems you want to have graded. Some possibly helpful formulas are given at the end of the exam.

1. An electron is at rest in an oscillating magnetic field $\mathbf{B} = B_0 \cos(\omega t)\hat{\mathbf{z}}$, where B_0 and ω are constants.

(a) [2] Construct the Hamiltonian matrix for this system.

The Hamiltonian due to spin is given by $H = ge\mathbf{B}\cdot\mathbf{S}/(2m)$, where g is the gyromagnetic ratio (g = 2.002…), and e is the fundamental charge. Keeping in mind that $S_z = \frac{1}{2}\hbar\sigma_z$, and σ_z is diagonal in the usual basis, we have

$$H = \frac{geB_0}{2m}\cdot\frac{\hbar}{2}\sigma_x \cos(\omega t) = \frac{geB_0\hbar}{4m}\cos(\omega t)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(b) [10] The electron starts out (at $t = 0$) in the spin-up state with respect to the x-axis (that is, $|\Psi(0)\rangle = |+_x\rangle$). Determine $|\Psi(t)\rangle$ at subsequent times.

The state with spin in the $+x$-direction is given in this basis by $|\Psi(0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. This is our boundary condition. Schrödinger's equation is

$$-i\hbar\frac{d}{dt}|\Psi(t)\rangle = H|\Psi(t)\rangle = \frac{geB_0\hbar}{4m}\cos(\omega t)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}|\Psi(t)\rangle.$$

If we write the two components of $|\Psi(t)\rangle$ as a and b, for example, this would be

$$\frac{d}{dt}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{geB_0 i}{4m}\cos(\omega t)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{geB_0 i}{4m}\cos(\omega t)\begin{pmatrix} a \\ -b \end{pmatrix}.$$

This obviously splits into two nearly identical equations, namely

$$\frac{da}{dt} = i\frac{geB_0}{4m}\cos(\omega t)a \quad \text{and} \quad \frac{db}{dt} = -\frac{geB_0 i}{4m}\cos(\omega t)b.$$

Multiplying the first equation by dt/a and the second by dt/b and then integrating, we have

$$\int\frac{da}{a} = i\frac{geB_0}{4m}\int\cos(\omega t)\,dt \quad \text{and} \quad \int\frac{db}{b} = -i\frac{geB_0}{4m}\int\cos(\omega t)\,dt,$$

$$\ln(a) = \frac{igeB_0}{4m\omega}\sin(\omega t) + c_a \quad \text{and} \quad \ln(b) = -\frac{igeB_0}{4m\omega}\sin(\omega t) + c_b.$$

Exponentiating these two expressions, and making sure the initial values match at $t = 0$, we can solve for a and b and find

$$\left|\Psi(t)\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \exp\left[igEB_0 \sin(\omega t)/4m\omega\right] \\ \exp\left[-igEB_0 \sin(\omega t)/4m\omega\right] \end{pmatrix}.$$

(c) [6] Find the probability of getting $-\frac{1}{2}\hbar$ if you measure S_x at time t.

To get the probability, we have to find $\left|\left\langle -_y \middle| \Psi(t)\right\rangle\right|^2$, which is

$$P(-_y) = \left|\left\langle -_y \middle| \Psi(t)\right\rangle\right|^2 = \left|\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} \exp\left[igeB_0 \sin(\omega t)/4m\omega\right] \\ \exp\left[-igeB_0 \sin(\omega t)/4m\omega\right] \end{pmatrix}\right|^2$$

$$= \frac{1}{4}\left|\exp\left[igeB_0 \sin(\omega t)/4m\omega\right] - \exp\left[-igeB_0 \sin(\omega t)/4m\omega\right]\right|^2$$

$$= \frac{1}{4}\left|2i\sin\left[geB_0 \sin(\omega t)/4m\omega\right]\right|^2 = \sin^2\left[geB_0 \sin(\omega t)/4m\omega\right].$$

(d) [7] What is the minimum field (B_0) required to force a complete flip in S_x?

To get it to flip with 100% certainty, we need the argument of the outer sine function to at least reach $\pm\frac{1}{2}\pi$. Since the inner sine function never exceeds ± 1, that means that the rest of the expression must exceed $\frac{1}{2}\pi$, so $geB_0/4m\omega \geq \frac{1}{2}\pi$, or in summary,

$$B_0 \geq \frac{2\pi m\omega}{ge}.$$

Note that if we approximate $g = 2$, we can simplify this a bit.

2. A particle of mass *m* is in the ground state of an infinite square of length *a*, with allowed region $0 \le x \le a$. Suddenly, the walls expand so that the right wall is now at *x* = 2*a* (and the left wall stays in the same place).

(a) [5] What is the probability that, immediately after the wall has expanded, the particle will be found between (i) $0 \le x \le a$, and (ii) $a \le x \le 2a$?

In the sudden approximation, the wave function doesn't change when you move the walls. Since the wave function was entirely in the first region, it will still be there, so the probabilities in the two cases are (i) 100% and (ii) 0%.

(b) [15] What is the probability that if you measured the energy you would get $E = \dfrac{\pi^2\hbar^2}{8ma^2}$?

For this we need the eigenstates of both the initial and final infinite square well. For the initial infinite square well, we have

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi n x}{a}\right), \quad E_n = \frac{\pi^2\hbar^2}{2ma^2}$$

For the final infinite square well, the wave functions and energies are the same, except we substitute $a \to 2a$. The ground state is $n = 1$. Simply substituting $a \to 2a$, we see that we are asked if it is still in the ground state. The probability of going from the ground state to some other state in the sudden approximation is

$$P\left(|\psi_1\rangle \to |\psi'_n\rangle\right) = \left|\langle \psi'_n | \psi_1\rangle\right|^2$$

$$= \left|\sqrt{\frac{2}{a}}\sqrt{\frac{2}{2a}}\int_0^a \sin\left(\frac{\pi n x}{2a}\right)\sin\left(\frac{\pi x}{a}\right)dx\right|^2 = \frac{2}{a^2}\left|\int_0^a \sin\left(\frac{\pi n x}{2a}\right)\sin\left(\frac{\pi x}{a}\right)dx\right|^2.$$

The integrals were only performed up to *a* because the initial wave function cuts off there. In particular, the probabilities for going to two lowest energy states are

$$P\left(|\psi_1\rangle \to |\psi'_1\rangle\right) = \frac{2}{a^2}\left|\int_0^a \sin\left(\frac{\pi x}{2a}\right)\sin\left(\frac{\pi x}{a}\right)dx\right|^2 = \frac{2}{a^2}\left|\frac{1}{2}\int_0^a\left[\cos\left(\frac{\pi x}{2a}\right) - \cos\left(\frac{\pi x}{2a}\right)\right]dx\right|^2$$

$$= \frac{1}{2a^2}\left|\left[\frac{2a}{\pi}\sin\left(\frac{\pi x}{2a}\right) - \frac{2a}{3\pi}\sin\left(\frac{\pi x}{2a}\right)\right]_0^a\right|^2 = \frac{1}{2a^2}\left|\frac{2a}{\pi} + \frac{2a}{3\pi}\right|^2 = \frac{1}{2a^2}\left(\frac{8a}{3\pi}\right)^2 = \frac{32}{9\pi^2},$$

$$P\left(|\psi_1\rangle \to |\psi'_2\rangle\right) = \frac{2}{a^2}\left|\int_0^a \sin^2\left(\frac{\pi x}{a}\right)dx\right|^2 = \frac{2}{a^2}\left|\frac{1}{2}\int_0^a\left[1 - \cos\left(\frac{2\pi x}{a}\right)\right]dx\right|$$

$$= \frac{1}{2a^2}\left|\left[x - \frac{a}{2\pi}\sin\left(\frac{2\pi x}{a}\right)\right]_0^a\right| = \frac{1}{2a^2}|a|^2 = \frac{1}{2}.$$

If converted to probabilities, these work out to 36% and 50% respectively, so it is clear that these two are the most likely cases, since all others add up to less than 14%.

(c) [5] Assume that you got the energy given in part (b). If the right wall is now slowly returned to $x = a$, what would be the most likely result of an energy measurement, and what would be the probability of obtaining that result?

As the wall is slowly moved back, we can use the adiabatic approximation, which says that the n'th eigenstate goes to the n'th eigenstate. In particular, the ground state will go to the ground state, so the final energy will be $E_1 = \dfrac{\pi^2\hbar^2}{2ma^2}$, with a probability of 100%.

3. A particle of mass m lies in a three-dimensional attractive power-law potential, $V(r) = -Ar^{-3/2}$. Using the variational principle, estimate the energy of the ground state using the unnormalized trial wave function $\psi(\mathbf{r}) = e^{-\lambda r}$. Can we be sure that this potential actually has a bound state?

We start by calculating the normalization integral $\langle\psi|\psi\rangle$, as well as the expectation values of the potential and the kinetic term. We have

$$\langle\psi|\psi\rangle = \int|\psi(r)|^2\,d^3\mathbf{r} = \int d\Omega\int_0^\infty e^{-2\lambda r}r^2dr = 4\pi\frac{\Gamma(3)}{(2\lambda)^3} = \frac{\pi}{\lambda^3},$$

$$\langle\psi|P^2|\psi\rangle = -\hbar^2\int\psi^*(r)\nabla^2\psi(r)\,d^3\mathbf{r} = -\hbar^2\int d\Omega\int_0^\infty e^{-\lambda r}\left(\frac{\partial^2}{\partial r^2}e^{-\lambda r} + \frac{2}{r}\frac{\partial}{\partial r}e^{-\lambda r}\right)r^2dr$$

$$= -4\pi\hbar^2\int_0^\infty e^{-2\lambda r}\left(\lambda^2r^2 - 2\lambda r\right)dr = -4\pi\hbar^2\left[\lambda^2\frac{\Gamma(3)}{(2\lambda)^3} - 2\lambda\frac{\Gamma(2)}{(2\lambda)^2}\right]$$

$$= 4\pi\hbar^2\left(\frac{1}{2\lambda} - \frac{1}{4\lambda}\right) = \frac{\pi\hbar^2}{\lambda},$$

$$\langle\psi|V|\psi\rangle = \int\psi^*(r)V(r)\psi(r)\,d^3\mathbf{r} = -A\int d\Omega\int_0^\infty r^{-3/2}r^2e^{-2\lambda r}dr = -\frac{4\pi A\Gamma\left(\frac{3}{2}\right)}{(2\lambda)^{3/2}} = -\frac{\pi A\sqrt{\pi}}{\lambda^{3/2}\sqrt{2}}.$$

We now put it all together to get the expectation value of the energy E as a function of the variational parameter λ:

$$E(\lambda) = \frac{\langle H\rangle}{\langle\psi|\psi\rangle} = \frac{1}{\langle\psi|\psi\rangle}\left[\frac{\langle P^2\rangle}{2m} + \langle V\rangle\right] = \frac{\lambda^3}{\pi}\left[\frac{\pi\hbar^2}{2m\lambda} - \frac{\pi A\sqrt{\pi}}{\lambda^{3/2}\sqrt{2}}\right] = \frac{\hbar^2\lambda^2}{2m} - A\lambda^{3/2}\sqrt{\frac{\pi}{2}}.$$

We need to find the minimum of this function, which we do by setting the derivative to zero, which yields

$$0 = \frac{d}{d\lambda} E(\lambda) = \frac{\hbar^2 \lambda}{m} - \frac{3}{2} A \lambda^{1/2} \sqrt{\frac{\pi}{2}},$$

$$\sqrt{\lambda} = \frac{3\sqrt{\pi} m A}{2\sqrt{2}\hbar^2}, \quad \lambda = \frac{9\pi A^2 m^2}{8\hbar^4}.$$

Substituting this back into the energy, we get an estimate of the ground state energy,

$$E(\lambda_{\text{min}}) = \frac{\hbar^2}{2m}\left(\frac{9\pi A^2 m^2}{8\hbar^4}\right)^2 - A\left(\frac{9\pi A^2 m^2}{8\hbar^4}\right)^{3/2}\sqrt{\frac{\pi}{2}} = \frac{81\pi^2 A^4 m^3}{128\hbar^6} - \frac{27\pi^2 A^4 m^3}{32\hbar^6} = -\frac{27\pi^2 A^4 m^3}{128\hbar^6}.$$

The actual energy must be smaller than this (more negative), and hence there must truly be a bound state, even if this isn't exactly the right energy.

4. A particle of mass *m* lies in a symmetric 2D harmonic oscillator with potential $V = \frac{1}{2} m\omega^2 (x^2 + y^2)$. In addition, there is a small perturbation $W = \gamma L_z = \gamma (x p_y - y p_x)$.

(a) [4] Find the eigenstates and energies of the unperturbed Hamiltonian. You do not need to give explicit forms for these eigenstates, it is sufficient to simply label them as, say, $|n_x, n_y\rangle$. Check that the ground state is non-degenerate, but the first excited state is degenerate.

Since this is just the sum of two ordinary Harmonic oscillators, the states are indeed of the form $|n_x, n_y\rangle$ and have energy $E_{n_x n_y} = \hbar\omega(n_x + \frac{1}{2}) + \hbar\omega(n_y + \frac{1}{2}) = \hbar\omega(n_x + n_y + 1)$ The ground state is $|0,0\rangle$ with energy $\hbar\omega$, and the first excited states are $|1,0\rangle$ and $|0,1\rangle$ with energy $2\hbar\omega$.

(b) [9] Show that the ground state is an exact eigenstate of the perturbed Hamiltonian.

It is already an exact eigenstate of the unperturbed Hamiltonian. The perturbation can be written in terms of raising and lowering operators as

$$W = \gamma(x p_y - y p_x) = i\gamma \sqrt{\frac{\hbar}{2m\omega}} \sqrt{\frac{\hbar m\omega}{2}} \left[(a_x + a_x^\dagger)(a_y^\dagger - a_y) - (a_y + a_y^\dagger)(a_x^\dagger - a_x) \right]$$

$$= \frac{1}{2} i\gamma\hbar \left(2a_y^\dagger a_x - 2a_x^\dagger a_y \right) = i\gamma\hbar \left(a_y^\dagger a_x - a_x^\dagger a_y \right).$$

It is then obvious that $W|0,0\rangle = i\gamma\hbar\left(a_y^\dagger a_x - a_x^\dagger a_y\right)|0,0\rangle = 0$.

(c) [12] Find the energies of the first excited states to first order in *γ*, and the corresponding eigenstates to leading order.

We need to find all matrix elements of the form $\langle a|W|b\rangle$ using these two states. We first see that

$$W|1,0\rangle = i\gamma\hbar\left(a_y^\dagger a_x - a_x^\dagger a_y\right)|1,0\rangle = i\gamma\hbar a_y^\dagger a_x|1,0\rangle = i\gamma\hbar|0,1\rangle,$$
$$W|0,1\rangle = i\gamma\hbar\left(a_y^\dagger a_x - a_x^\dagger a_y\right)|0,1\rangle = -i\gamma\hbar a_x^\dagger a_y|0,1\rangle = -i\gamma\hbar|1,0\rangle.$$

We now write the perturbation matrix as

$$\tilde{W} = \begin{pmatrix} \langle 10|W|10\rangle & \langle 10|W|01\rangle \\ \langle 01|W|10\rangle & \langle 01|W|01\rangle \end{pmatrix} = \hbar\gamma\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

The final matrix is the Pauli matrix σ_y and has eigenvalues ± 1. The corresponding eigenvectors can be found by solving

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \pm\begin{pmatrix} a \\ b \end{pmatrix},$$
$$ib = \mp a \quad \text{and} \quad ia = \pm b.$$

So $b = \pm ia$, and if we normalize them then $1 = |a|^2 + |b|^2 = 2|a|^2$, we have states $|\pm\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm i \end{pmatrix}$.

This gives us our perturbed states and eigenvalues

$$|\pm\rangle = \tfrac{1}{\sqrt{2}}\left(|1,0\rangle \pm i|0,1\rangle\right), \quad \text{with energy} \quad E_\pm = 2\hbar\omega \pm \hbar\gamma.$$

Incidentally, since these two states are not mixed by W at all with any other energy states, these eigenstates and eigenvalues are exact as well.

5. A measurement corresponding to observable *A*, has two normalized eigenstates $|\psi_1\rangle$ and $|\psi_2\rangle$ with eigenvalues a_1 and a_2 respectively. A second measurement corresponding to observable *B*, has two normalized eigenstates $|\phi_1\rangle$ and $|\phi_2\rangle$ with eigenvalues b_1 and b_2 respectively. The eigenstates are related by

$$|\psi_1\rangle = \tfrac{3}{5}|\phi_1\rangle + \tfrac{4}{5}|\phi_2\rangle \quad ; \quad |\psi_2\rangle = \tfrac{4}{5}|\phi_1\rangle - \tfrac{3}{5}|\phi_2\rangle$$

(a) [8] Observable *A* is measured, and the value a_1 is obtained. What is the state of the wave function $|\Psi\rangle$ immediately after this measurement?

When a measurement is performed, the state vector will collapse into an eigenstate of the corresponding operator, with eigenvalue equal to the measured quantity. Since we obtained the eigenvalue a_1 the wave function must be in an eigenstate of A with this eigenvalue a_1. This state is uniqualy determined, up to an irrelevant phase, to be the state $|\psi_1\rangle$, so we must have

$|\Psi\rangle = |\psi_1\rangle$

(b) [8] If *B* is now measured, what are the possible results, and what are their probabilities?

In general, the only possible outcome of a measurement is the eigenvalues of the corresponding operator, so the outcome must be b_1 or b_2. The corresponding probability is just the square of the amplitude of the corresponding wave functions, we

$$P(b_1) = \left|\langle \phi_1 | \Psi \rangle\right|^2 = \left|\langle \phi_1 | \psi_1 \rangle\right|^2 = \left|\tfrac{3}{5}\langle \phi_1 | \phi_1 \rangle + \tfrac{4}{5}\langle \phi_1 | \phi_2 \rangle\right|^2 = \left|\tfrac{3}{5}\right|^2 = \tfrac{9}{25},$$

$$P(b_2) = \left|\langle \phi_2 | \Psi \rangle\right|^2 = \left|\langle \phi_2 | \psi_1 \rangle\right|^2 = \left|\tfrac{3}{5}\langle \phi_2 | \phi_1 \rangle + \tfrac{4}{5}\langle \phi_2 | \phi_2 \rangle\right|^2 = \left|\tfrac{4}{5}\right|^2 = \tfrac{16}{25}.$$

Depending on the outcome of the measurement, the system will now be in the quantum state $|\phi_1\rangle$ or $|\phi_2\rangle$ if the measurement yielded b_1 or b_2 respectively.

(c) [9] Assume in part (b) that the result yielded result b_1. Right after the measurement of B, A is measured again. What is the probability of getting the value a_1 again? How would your answer be different if the measurement in part (b) had been b_2 instead?

If the *B* measurement had yielded b_1, then the wave function would now be $|\phi_1\rangle$, and the probability would be

$$P(a_1|b_1) = \left|\langle \psi_1 | \Psi \rangle\right|^2 = \left|\langle \psi_1 | \phi_1 \rangle\right|^2 = \left|\tfrac{3}{5}\langle \phi_1 | \phi_1 \rangle + \tfrac{4}{5}\langle \phi_2 | \phi_1 \rangle\right|^2 = \left|\tfrac{3}{5}\right|^2 = \tfrac{9}{25}.$$

Similarly, if the measurement of *B* had yielded b_2, then the probability would be

$$P(a_1|b_2) = \left|\langle \psi_1 | \Psi \rangle\right|^2 = \left|\langle \psi_1 | \phi_2 \rangle\right|^2 = \left|\tfrac{3}{5}\langle \phi_1 | \phi_1 \rangle + \tfrac{4}{5}\langle \phi_2 | \phi_2 \rangle\right|^2 = \left|\tfrac{4}{5}\right|^2 = \tfrac{16}{25}.$$

We can also address the question of what the probability of outcome a_1 would be if we didn't know the outcome of the first measurement, though we weren't asked that.

$$P(a_1) = P(b_1)P(a_1|b_1) + P(b_2)P(a_1|b_2) = \left(\tfrac{9}{25}\right)^2 + \left(\tfrac{16}{25}\right)^2 = \tfrac{81}{625} + \tfrac{256}{625} = \tfrac{337}{625}.$$