Graduate Quantum Mechanics

Solutions to Chapter 14

1. [15] This problem has nothing to do with quantum mechanics. A rigid sphere of radius *xa* is launched at a stationary sphere of radius *a*, with *x* < 1. The two spheres have the same density ρ. In the center of mass frame, the differential cross section is given by $d\sigma/d\Omega = \frac{1}{4}a^2(x+1)^2$. This differential cross-section is independent of both the velocity and the angle of deflection.

(a) [2] What is the total cross section measured in the lab frame? (this is easy)

The lab frame and center of mass frame total cross section are identical. To find it, simply integrate the differential cross-section over solid angle, which is the same as multiplying by 4π. The total cross-section is therefore $\sigma = \pi a^2(1+x)^2$.

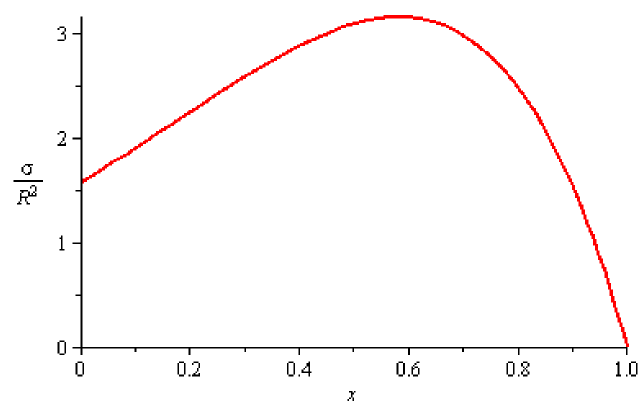
(b) [4] As a function of *x*, for what center of mass frame angles θ will the lab frame scattering be by an angle greater than 90 degrees, so $\theta_{\text{lab}} > 90°$?

The two angles are related by

$$\tan\theta_L = \frac{\sin\theta}{\gamma + \cos\theta}$$

To get the lab angle to be bigger than 90 degrees, we need the right side to be negative, which implies $\cos\theta + \gamma < 0$, or $\theta > \cos^{-1}(-\gamma)$. γ is the ratio of masses, which, since they have the same density, is the same as the ratio of volumes, which clearly is $\gamma = (xa)^3/a^3 = x^3$, so $\theta > \cos^{-1}(-x^3)$.

(c) [6] Find a formula for the cross-section, but restricted to angles exceeding 90 degrees in the lab. Sketch its behavior (by hand or with a computer) for the entire range $0 < x < 1$.



We simply work out the corresponding integral, which is straightforward:

$$\sigma(\theta_l > 90°) = \int_0^{2\pi} d\phi \int_{-1}^{-\gamma} \frac{d\sigma}{d\Omega} d\cos\theta = \tfrac{1}{4}(2\pi)(xa+a)^2(1-\gamma) = \tfrac{1}{2}\pi a^2(1+x)^2(1-x^3)$$

We plot it with the help of Maple (above).

(d) [3] For what value of x is the restricted cross section you found in part (c) the largest? You can do this numerically, if necessary.

Obviously, there is a peak somewhere around $x = 0.6$. To find it, take the derivative and set it to zero.

$$0 = \frac{d}{dx}\sigma\left(\theta_l > 90°\right) = \tfrac{1}{2}\pi a^2\left[2\left(1+x\right)\left(1-x^3\right) - 3x^2\left(1+x\right)^2\right] = \tfrac{1}{2}\pi a^2\left(1+x\right)\left[2-3x^2-5x^3\right]$$

We can let Maple solve it exactly, or with a bit of work we can actually solve the cubic.

$$x = \frac{2}{\sqrt[3]{10+\sqrt{92}}+\sqrt[3]{10-\sqrt{92}}} = 0.581783$$

2. [5] Find a formula for the center of mass angle θ in terms of the lab angle θ_{lab}. For what set of masses m and M is there a maximum deflection angle $\theta_{\max}$ by which an object can be deflected, as measured in the lab frame? Find a formula for $\theta_{\max}$ when this applies.

We start with the formula $\sin\left(\theta-\theta_L\right) = \gamma\sin\theta_L$, then it is just a couple of steps to write

$$\theta - \theta_L = \sin^{-1}\left(\gamma\sin\theta_L\right),$$
$$\theta = \theta_L + \sin^{-1}\left(\gamma\sin\theta_L\right).$$

An equivalent formula is

$$\theta = \cos^{-1}\left(-\gamma\sin^2\theta_L \pm \cos\theta_L\sqrt{1-\gamma^2\sin^2\theta_L}\right).$$

Either of these equations run into trouble if $\gamma\sin\theta_L > 1$. This implies you can't let this happen, so we must have

$$\gamma\sin\theta_L \le 1,$$
$$\theta_L \le \sin^{-1}\left(\frac{1}{\gamma}\right) = \sin^{-1}\frac{M}{m}.$$

This limit is only meaningful if $M \le m$, so the projectile mass equals or exceeds the target mass.