Physics 742 – Graduate Quantum Mechanics 2

Solutions to Chapter 15

5. [20] A hydrogen atom is in the 1s ground state while being bathed in light of sufficient frequency to excite it to the n = 2 states. The light is traveling in the +z direction and has circular polarization, $\boldsymbol{\varepsilon} = \frac{1}{\sqrt{2}}\left(\hat{\mathbf{x}} + i\hat{\mathbf{y}}\right)$.

(a) [10] Calculate all relevant dipole moments $\mathbf{r}_{FI}$ for final states $\left|2lm\right\rangle$.

The matrix elements we need are

$$\left\langle 2lm\right|\mathbf{R}\cdot\boldsymbol{\varepsilon}\left|100\right\rangle = \tfrac{1}{\sqrt{2}}\left\langle 2lm\right|\left(X+iY\right)\left|100\right\rangle = \tfrac{1}{\sqrt{2}}\int d^3\mathbf{r}\,\psi_{2lm}^*\left(\mathbf{r}\right)\psi_{100}\left(\mathbf{r}\right) r\sin\theta\left(\cos\phi + i\sin\phi\right)$$

$$= \tfrac{1}{\sqrt{2}}\int_0^\infty r^3 dr R_{2l}\left(r\right)R_{10}\left(r\right)\int d\Omega\, Y_l^m\left(\theta,\phi\right)^* Y_0^0\left(\theta,\phi\right)\sin\theta e^{i\phi}\,.$$

Of course, $Y_0^0 = 1\big/\sqrt{4\pi}$, and comparison of the final factors with spherical harmonics makes it clear that $\sin\theta e^{i\phi} = -Y_1^1\left(\theta,\phi\right)\sqrt{8\pi/3}$, so when we put this all together, we have

$$\left\langle 2lm\right|\mathbf{R}\cdot\boldsymbol{\varepsilon}\left|100\right\rangle = -\frac{1}{\sqrt{2}}\frac{1}{\sqrt{4\pi}}\sqrt{\frac{8\pi}{3}}\int_0^\infty r^3 dr R_{2l}\left(r\right)R_{10}\left(r\right)\int d\Omega\, Y_l^m\left(\theta,\phi\right)^* Y_1^1\left(\theta,\phi\right)$$

$$= \frac{-1}{\sqrt{3}}\delta_{l1}\delta_{m1}\int_0^\infty r^3 dr R_{21}\left(r\right)R_{10}\left(r\right) = \frac{-2}{\sqrt{3\cdot 24 a_0^5 a_0^3}}\delta_{l1}\delta_{m1}\int_0^\infty r^3 dr\left(re^{-r/2a_0}\right)e^{-r/a_0}$$

$$= \frac{-1}{3\sqrt{2}a_0^4}\delta_{l1}\delta_{m1}4!\left(\frac{2}{3}a_0\right)^5 = -\frac{128\sqrt{2}}{243}a_0\delta_{l1}\delta_{m1}\,.$$

Doing it by hand seemed easier than using Maple. We used orthogonality of the Y's to do the angular integral.

(b) [4] Find a formula for the rate at which the atom makes this transition.

We now simply use equation (15.30) from the notes, to obtain

$$\Gamma\left(I\to F\right) = 4\pi^2\alpha\hbar^{-1}\mathcal{I}\left(\omega_{FI}\right)\left(\frac{128\sqrt{2}}{243}a_0\right)^2 = \frac{2^{17}\pi^2\alpha a_0^2}{3^{10}\hbar}\mathcal{I}\left(\omega_{FI}\right).$$

(c) [6] What is the wavelength required for this transition? Assume at this wavelength the power is $\mathcal{I}\left(\lambda\right) = 100\ \text{W/m}^2/\text{nm}$. Find the rate at which the atom converts. (Note the footnote on p. 280)

. The energy involved is $E = \frac{3}{4}\left(13.60\ \text{eV}\right) = 10.20\ \text{eV}$. We can then calculate the relevant wavelength from

$$\lambda = \frac{c}{f} = \frac{hc}{E} = \frac{1240\ \text{nm}\cdot\text{eV}}{10.20\ \text{eV}} = 121.5\ \text{nm}\,.$$

We then calculate the intensity per unit angular frequency as

$$\mathcal{I}(\omega) = \frac{\lambda^2}{2\pi c}\mathcal{I}(\lambda) = \frac{(121.5\ \text{nm})^2 (100\ \text{W/m}^2/\text{nm})}{2\pi (2.998\times 10^8\ \text{m/s})(10^9\ \text{nm/m})} = 7.84\times 10^{-13}\ \text{W}\cdot\text{s/m}^2\,.$$

Substituting this into our expression, we find

$$\Gamma\left(|100\rangle \to |211\rangle\right) = \frac{2^{17}\pi^2 (5.29\times 10^{-11}\ \text{m})^2 (7.84\times 10^{-13}\ \text{J/m}^2)}{3^{10}(137)(1.054\times 10^{-34}\ \text{J}\cdot\text{s})} = 3.33\ \text{s}^{-1}\,.$$

So even with this rather weak source, an atom will undergo such a transition several times per second. The reverse rate *would* be the same, except it can be shown that spontaneous emission, a process we have not yet accounted for, is a much faster process.