Solutions to Chapter 15

6. [25] A hydrogen atom is in interstellar space in the 1s state, but not in the true ground state (F = 0), but rather in the hyperfine excited state (F=1), specifically in the state $\left|\phi_I\right\rangle = \left|n,l,j,F,m_F\right\rangle = \left|1,0,\tfrac{1}{2},1,0\right\rangle$. It is going to transition to the true ground state $\left|\phi_F\right\rangle = \left|n,l,j,F,m_F\right\rangle = \left|1,0,\tfrac{1}{2},0,0\right\rangle$ via a magnetic dipole interaction.

(a) [5] Write out the initial and final states in terms of the explicit spin states of the electron and proton $\left|\pm,\pm\right\rangle$. Find all non-zero components of the matrix $\left\langle\phi_F\right|\mathbf{S}\left|\phi_I\right\rangle$, where S is the electron spin operator.

The explicit form for these spin states can be found, for example, eq. (8.21). We therefore have

$$\left|1,0\right\rangle = \tfrac{1}{\sqrt{2}}\left(\left|+-\right\rangle + \left|-+\right\rangle\right), \quad \left|0,0\right\rangle = \tfrac{1}{\sqrt{2}}\left(\left|+-\right\rangle - \left|-+\right\rangle\right).$$

The spin operator acting on the initial state yields

$$S_x\left|10\right\rangle = \tfrac{1}{2}\hbar\sigma_x\tfrac{1}{\sqrt{2}}\left(\left|+-\right\rangle+\left|-+\right\rangle\right) = \tfrac{1}{\sqrt{8}}\hbar\left(\left|--\right\rangle+\left|++\right\rangle\right),$$
$$S_y\left|10\right\rangle = \tfrac{1}{2}\hbar\sigma_y\tfrac{1}{\sqrt{2}}\left(\left|+-\right\rangle+\left|-+\right\rangle\right) = \tfrac{1}{\sqrt{8}}\hbar\left(i\left|--\right\rangle-i\left|++\right\rangle\right),$$
$$S_z\left|10\right\rangle = \tfrac{1}{2}\hbar\sigma_z\tfrac{1}{\sqrt{2}}\left(\left|+-\right\rangle+\left|-+\right\rangle\right) = \tfrac{1}{\sqrt{8}}\hbar\left(\left|+-\right\rangle-\left|-+\right\rangle\right) = \tfrac{1}{2}\hbar\left|00\right\rangle.$$

Therefore, the only non-zero matrix element is S_z, so $\left\langle\phi_F\right|\mathbf{S}\left|\phi_I\right\rangle = \tfrac{1}{2}\hbar\hat{\mathbf{z}}$.

(b) [8] Show that the rate for this transition for a wave going in a specific direction with a definite polarization is given by $\Gamma = 4\pi^2 m^{-2}\omega^{-2}\alpha\mathcal{I}\left|(\mathbf{k}\times\boldsymbol{\varepsilon})\cdot\mathbf{S}_{FI}\right|^2\delta\left(E_F - E_I + \hbar\omega\right)$.

Our starting point is the equation in the middle of page 283:

$$W_{FI} = A_0 e\left\{-\omega_{FI}\sum_i\left\langle\phi_F\right|\left(\mathbf{k}\cdot\mathbf{R}_i\right)\left(\boldsymbol{\varepsilon}\cdot\mathbf{R}_i\right)\left|\phi_I\right\rangle + \frac{i}{m}\left(\mathbf{k}\times\boldsymbol{\varepsilon}\right)\cdot\left\langle\phi_F\right|\left(\tfrac{1}{2}\mathbf{L}+\mathbf{S}\right)\left|\phi_I\right\rangle\right\}$$

The first term won't contribute in this case, since the operator can't connect the ground state with the ground state. The operator **L** vanishes on an s-electron, so the only relevant term is

$$W_{FI} = A_0 e\frac{i}{m}\left(\mathbf{k}\times\boldsymbol{\varepsilon}\right)\cdot\left\langle\phi_F\right|\mathbf{S}\left|\phi_I\right\rangle = A_0 e\frac{i}{m}\left(\mathbf{k}\times\boldsymbol{\varepsilon}\right)\cdot\mathbf{S}_{FI}$$

We now technically need to use equation (15.22) (with the roles of I and F reversed) to give

$$\Gamma\left(I\to F\right) = \frac{2\pi}{\hbar}\left|W_{FI}^\dagger\right|^2\delta\left(E_F - E_I + \hbar\omega\right) = \frac{2\pi A_0^2 e^2}{\hbar m^2}\left|\left(\mathbf{k}\times\boldsymbol{\varepsilon}\right)\cdot\mathbf{S}_{FI}\right|^2\delta\left(E_F - E_I + \hbar\omega\right)$$

In a manner very similar to the notes, we use equation (15.28), together with $\alpha = k_e e^2/\hbar c$ to rewrite this as

$$\Gamma(I\to F)=\frac{4\pi^2\mathcal{I}k_e e^2}{\hbar m^2\omega^2 c}\left|(\mathbf{k}\times\boldsymbol{\varepsilon})\cdot\mathbf{S}_{FI}\right|^2\delta(E_F-E_I+\hbar\omega)=\frac{4\pi^2\alpha\mathcal{I}}{m^2\omega^2}\left|(\mathbf{k}\times\boldsymbol{\varepsilon})\cdot\mathbf{S}_{FI}\right|^2\delta(E_F-E_I+\hbar\omega)$$

(c) [7] Show that for a wave going in a random direction with random polarization, this simplifies to $\Gamma(I\to F)=\frac{4}{3}\pi^2\alpha\mathcal{I}\left|\mathbf{S}_{FI}\right|^2\delta(E_f-E_i+\hbar\omega)/m^2c^2$.

To simplify, let's assume that $\mathbf{S}_{FI}$ is in the *z*-direction, and the direction of **k** is at an angle θ compared to $\mathbf{S}_{FI}$. The simplest way to tackle this is to first rewrite the triple product as

$$(\mathbf{k}\times\boldsymbol{\varepsilon})\cdot\mathbf{S}_{FI}=(\mathbf{S}_{FI}\times\mathbf{k})\cdot\boldsymbol{\varepsilon}.$$

Now, the cross product is perpendicular to both $\mathbf{S}_{FI}$ and **k**, and has magnitude

$$\left|\mathbf{S}_{FI}\times\mathbf{k}\right|=\left|\mathbf{S}_{FI}\right|k\sin\theta$$

The polarization must be perpendicular to **k**. One polarization is in the same plane as $\mathbf{S}_{FI}$ and **k**, and this will give no contribution to $(\mathbf{S}_{FI}\times\mathbf{k})\cdot\boldsymbol{\varepsilon}$. The other will be in the same direction as $\mathbf{S}_{FI}\times\mathbf{k}$, so $(\mathbf{S}_{FI}\times\mathbf{k})\cdot\boldsymbol{\varepsilon}=\left|\mathbf{S}_{FI}\right|k\sin\theta$ (up to an arbitrary sign). Hence we have

$$\Gamma_{\text{unpol}}=\frac{1}{2}\sin^2\theta\frac{4\pi^2\alpha\mathcal{I}k^2}{m^2\omega^2}\left|\mathbf{S}_{FI}\right|^2\delta(E_f-E_i+\hbar\omega)=\frac{2\pi^2\alpha\mathcal{I}}{m^2c^2}\left|\mathbf{S}_{FI}\right|^2\delta(E_f-E_i+\hbar\omega)\sin^2\theta.$$

We now must average this over all angles, which gives us a factor of

$$\left\langle\sin^2\theta\right\rangle=\int\frac{d\Omega}{4\pi}\sin^2\theta=\frac{1}{4\pi}\int_0^{2\pi}d\phi\int_{-1}^{1}\left(1-\cos^2\theta\right)d\cos\theta=\frac{1}{2}\left(\cos\theta-\frac{1}{3}\cos^2\theta\right)\Bigg|_{-1}^{1}=\frac{2}{3}.$$

for a final answer of

$$\Gamma_{\text{random}}=\frac{4\pi^2\alpha\mathcal{I}}{3m^2c^2}\left|\mathbf{S}_{FI}\right|^2\delta(E_f-E_i+\hbar\omega).$$

(d) [5] For low frequencies, the cosmic microwave background intensity is $\mathcal{I}(\omega)=k_BT\omega^2/\pi^2c^2$ where k_B is Boltzman's constant and T is the temperature. Integrate the flipping rate over frequency. Find the mean time Γ^{-1} for a hydrogen atom to reverse itself in a background temperature T = 2.73 K for $\omega_{FI}=-2\pi(1.420\text{ GHz})$.

First, we take a factor of $\hbar$ out of the delta function to yield

$$\Gamma_{\text{random}}=\frac{4\pi^2\alpha\mathcal{I}}{3m^2c^2\hbar}\left|\mathbf{S}_{FI}\right|^2\delta(\omega_{FI}+\omega).$$

Obviously, the frequency we need is $\omega = -\omega_{FI} = 2\pi(1.420\text{ GHz})$. If we have a continuous distribution of intensities $\mathcal{I}(\omega)$, this changes to

$$\Gamma_{\text{random}} = \frac{4\pi^2\alpha}{3m^2c^2\hbar}\left|\mathbf{S}_{FI}\right|^2 \mathcal{I}\left(\left|\omega_{FI}\right|\right).$$

Substituting our various formulas in, and then making numerical substitutions, we find

$$\begin{aligned}\Gamma_{\text{random}} &= \frac{4\pi^2\alpha}{3m^2c^2\hbar}\left(\frac{\hbar}{2}\right)^2 \frac{k_B T\omega^2}{\pi^2c^2} = \frac{k_B T\alpha\hbar\omega^2}{3m^2c^4} \\ &= \frac{\left(1.381\times10^{-23}\text{ J/K}\right)\left(2.73\text{ K}\right)\left(1.054\times10^{-34}\text{ J}\cdot\text{s}\right)\left(2\pi\left(1.420\times10^{9}\text{ s}^{-1}\right)\right)^2}{3\cdot137\left(9.1094\times10^{-31}\text{ kg}\right)^2\left(2.998\times10^{8}\text{ m/s}\right)^4} \\ &= 1.148\times10^{-13}\text{ s}^{-1} = 3.623\times10^{-6}\text{ y}^{-1}.\end{aligned}$$

Taking the reciprocal, the mean flipping time is 276,000 years.