Physics 742 – Graduate Quantum Mechanics 2

Solutions to Chapter 16

1. **[10] Starting with the free Dirac Eqn. (16.4), multiply it on the left by $\Psi^\dagger$. Then take the Hermitian conjugate of Eqn. (16.4) and multiply it on the right by Ψ and subtract them. If we define the probability density as $\rho = \Psi^\dagger\Psi$, show that we can obtain a conservation equatioin of the form $\partial\rho/\partial t + \nabla\cdot\mathbf{j} = 0$ for a suitable choice of the probability current j. What is j? Now repeat with the electromagnetic version of the Dirac Eqn. (16.14).**

Since we have to consider the presence of electromagnetic fields, we might as well do it from the start, and then we can just set the fields to zero in the case of the free Dirac equation. The Dirac equation with electromagnetic fields is given by $i\hbar(\partial\Psi/\partial t) = H\Psi$, with H given at the beginning of section 16C:

$$i\hbar\frac{\partial\Psi}{\partial t} = -i\hbar c\boldsymbol{\alpha}\cdot\nabla\Psi + ce\boldsymbol{\alpha}\cdot\mathbf{A}(\mathbf{r},t)\Psi - eU(\mathbf{r},t)\Psi + mc^2\beta\Psi\ ,$$

where we have put in the explicit form of the momentum operator **P**, as well as changing **R** to **r** since we are acting on a wave function. Take the Hermitian conjugate of this expression, keeping in mind that the matrices $\boldsymbol{\alpha}$ and β are Hermitian. We have

$$-i\hbar\frac{\partial\Psi^\dagger}{\partial t} = i\hbar c\left(\nabla\Psi^\dagger\right)\cdot\boldsymbol{\alpha} + ce\Psi^\dagger\boldsymbol{\alpha}\cdot\mathbf{A}(\mathbf{r},t) - eU(\mathbf{r},t)\Psi^\dagger + mc^2\Psi^\dagger\beta\ .$$

Multiply the original equation on the left by $\Psi^\dagger$ and the second equation on the right by Ψ to yield

$$i\hbar\Psi^\dagger\frac{\partial\Psi}{\partial t} = -i\hbar c\Psi^\dagger\boldsymbol{\alpha}\cdot\nabla\Psi + ce\Psi^\dagger\boldsymbol{\alpha}\cdot\mathbf{A}(\mathbf{r},t)\Psi - eU(\mathbf{r},t)\Psi^\dagger\Psi + mc^2\Psi^\dagger\beta\Psi\ ,$$

$$-i\hbar\frac{\partial\Psi^\dagger}{\partial t}\Psi = i\hbar c\left(\nabla\Psi^\dagger\right)\cdot\boldsymbol{\alpha}\Psi + ce\Psi^\dagger\boldsymbol{\alpha}\cdot\mathbf{A}(\mathbf{r},t)\Psi - eU(\mathbf{r},t)\Psi^\dagger\Psi + mc^2\Psi^\dagger\beta\Psi\ .$$

Subtract these to get a new equation

$$i\hbar\left[\Psi^\dagger\frac{\partial\Psi}{\partial t} + \frac{\partial\Psi^\dagger}{\partial t}\Psi\right] = -i\hbar c\left[\left(\nabla\Psi^\dagger\right)\cdot\boldsymbol{\alpha}\Psi + \Psi^\dagger\boldsymbol{\alpha}\cdot\nabla\Psi\right],$$

$$\frac{\partial}{\partial t}\left(\Psi^\dagger\Psi\right) + c\nabla\cdot\left(\Psi^\dagger\boldsymbol{\alpha}\Psi\right) = 0\ .$$

If we define $\rho = \Psi^\dagger\Psi$ and $\mathbf{J} = c\Psi^\dagger\boldsymbol{\alpha}\Psi$, then this is equivalent to $\partial\rho/\partial t + \nabla\cdot\mathbf{J} = 0$. Hence we see that this definition of the probability density and the probability current works, even if we have electromagnetic fields.