

Physics 742 – Graduate Quantum Mechanics 2
Solutions to Chapter 18

3. [15] An electron is trapped in a 3D harmonic oscillator potential, $H = \mathbf{P}^2/2m + \frac{1}{2}m\omega_0^2\mathbf{R}^2$. It is in the quantum state $|n_x, n_y, n_z\rangle = |0, 0, 2\rangle$. It is going to decay *directly* into the ground state $|0, 0, 0\rangle$
- (a) [1] Convince yourself that it cannot go there via the electric dipole transition. It can, however, go there via the electric quadrupole transition.

The position operators \mathbf{R} only raise or lower one of the three eigenvalues, and hence $\langle 0, 0, 0 | \mathbf{R} | 0, 0, 2 \rangle$ vanishes.

- (b) [3] Calculate every non-vanishing matrix element of the form $\langle 0, 0, 0 | R_i R_j | 0, 0, 2 \rangle$.

There is only one non-vanishing component, namely

$$\langle 0, 0, 0 | ZZ | 0, 0, 2 \rangle = \frac{\hbar}{2m\omega_0} \langle 0, 0, 0 | a_z^2 | 0, 0, 2 \rangle = \frac{\hbar}{\sqrt{2}m\omega_0}$$

- (c) [7] Calculate the polarized differential decay rate $d\Gamma_{\text{pol}}(002 \rightarrow 000)/d\Omega$ for this decay. This will require, among many other things, converting a sum to an integral in the infinite volume limit.

As found in the lecture notes, the decay rate for electric quadrupole is given by

$$\frac{d\Gamma}{d\sigma} = \frac{\alpha\omega_{IF}^3}{8\pi c^2} \left| \sum_j \langle \phi_F | (\mathbf{k} \cdot \mathbf{R}_j) (\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}^* \cdot \mathbf{R}_j) | \phi_I \rangle \right|^2 = \frac{\alpha\omega_{IF}^3}{8\pi c^2} |k_z \varepsilon_z^* \langle 0, 0, 0 | Z^2 | 0, 0, 2 \rangle|^2 = \frac{\alpha\hbar^2 \omega_{IF}^3 k_z^2 |\varepsilon_z|^2}{16\pi c^2 m^2 \omega_0^2}.$$

The energy difference between the states $|0, 0, 2\rangle$ and $|0, 0, 0\rangle$ is $2\hbar\omega_0$, and therefore $\omega_{IF} = 2\omega_0$, and $k_z = k \cos \theta = 2\omega_0 \cos \theta / c$. Substituting this in, we have

$$\frac{d\Gamma}{d\sigma} = \frac{\alpha\hbar^2 8\omega_0^3}{16\pi c^2 m^2 \omega_0^2} \cdot \frac{4\omega_0^2 \cos^2 \theta}{c^2} |\varepsilon_z|^2 = \frac{2\alpha\hbar^2 \omega_0^3}{\pi c^4 m^2} \cos^2 \theta |\varepsilon_z|^2.$$

- (d) [4] Sum it over polarizations and integrate it over angles to determine the total decay rate $\Gamma(002 \rightarrow 000)$.

If we sum it over polarizations, the only polarization that will contribute is the one partially along the z -axis, for which we find $\boldsymbol{\varepsilon}_z = \sin \theta$, so we have

$$\frac{d\Gamma}{d\Omega} = \frac{2\alpha\hbar^2\omega_0^3}{\pi m^2 c^4} \cos^2 \theta \sin^2 \theta.$$

Integrating this over ϕ just gives a factor of 2π , and then the remaining integral yields

$$\Gamma = \frac{4\alpha\hbar^2\omega_0^3}{m^2 c^4} \int_0^\pi \cos^2 \theta \sin^2 \theta \sin \theta d\theta = \frac{16\alpha\hbar^2\omega_0^3}{15m^2 c^4}.$$