Physics 741 – Graduate Quantum Mechanics 1

Solutions to Chapter 1

1. [10] A particle of mass m lies in one-dimension in a potential of the form $V(x) = Fx$, where F is constant. The wave function at time t is given by

$$\Psi(x,t) = N(t)\exp\left[-\tfrac{1}{2}A(t)x^2 + B(t)x\right]$$

where N, A, and B are all complex functions of time. Use Schrödinger's equation to derive equations for the time derivative of the three functions A, B, and N. You do not need to solve these equations.

We first work out the time derivative and two space derivatives.

$$\frac{\partial\psi}{\partial t} = \left(\frac{dN}{dt} - \frac{N}{2}\frac{dA}{dt}x^2 + N\frac{dB}{dt}x\right)e^{-Ax^2/2+Bx},$$

$$\frac{\partial^2\psi}{\partial x^2} = N\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}e^{-Ax^2/2+Bx}\right] = N\frac{\partial}{\partial x}\left[(-Ax+B)e^{-Ax^2/2+Bx}\right] = N\left[-A+(-Ax+B)^2\right]e^{-Ax^2/2+Bx}.$$

Now we simply substitute these results into Schrödinger's equation:

$$i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi + V(x)\psi,$$

$$i\hbar\left(\frac{dN}{dt} - \frac{N}{2}\frac{dA}{dt}x^2 + N\frac{dB}{dt}x\right)e^{-Ax^2/2+Bx} = -\frac{\hbar^2}{2m}N\left[-A+(-Ax+B)^2\right]e^{-Ax^2/2+Bx} + FxNe^{-Ax^2/2+Bx}.$$

Canceling the common exponential and dividing by $i\hbar N$, this simplifies to

$$\frac{1}{N}\frac{dN}{dt} - \frac{1}{2}\frac{dA}{dt}x^2 + \frac{dB}{dt}x = \frac{i\hbar}{2m}\left(A^2x^2 - 2ABx + B^2 - A\right) - \frac{iF}{\hbar}x.$$

This expression must be true at all positions x. The only way this can happen is if the coefficients of x^2, x, and the constant terms all match on the two sides of the equation. This implies

$$\frac{dA}{dt} = -\frac{i\hbar}{m}A^2, \quad \frac{dB}{dt} = -\frac{i\hbar}{m}AB - \frac{iF}{\hbar}, \quad \frac{dN}{dt} = \frac{i\hbar}{2m}N\left(B^2 - A\right).$$

The first of these is, in fact, pretty easy to solve, but the others are a bit trickier.

$$\frac{1}{A(t)} = \frac{1}{A_0} + \frac{i\hbar}{m}t.$$

2. **[10] For each of the wave functions in one dimension given below, *N* and *a* are positive real numbers. Determine the normalization constant *N* in terms of *a*, and determine the probability that a measurement of the position of the particle will yield *x* > *a*.**

(a) [4] $\psi(x) = N/(x+ia)$

$$1 = \int_{-\infty}^{\infty} \psi^*(x)\psi(x)\,dx = \int_{-\infty}^{\infty} \frac{N^2 dx}{(x+ia)(x-ia)} = \int_{-\infty}^{\infty} \frac{N^2 dx}{x^2+a^2} = \frac{N^2}{a}\tan^{-1}(x/a)\Big|_{-\infty}^{\infty} = \frac{\pi N^2}{a},$$

$$N = \sqrt{a/\pi},$$

$$P(x>a) = (N^2/a)\tan^{-1}(x/a)\Big|_a^{\infty} = \frac{\frac{1}{2}\pi - \frac{1}{4}\pi}{\pi} = \tfrac{1}{4} = 25\%.$$

(b) [3] $\psi(x) = N\exp(-|x|/a)$

$$1 = N^2\int_{-\infty}^{\infty} e^{-2|x|/a}dx = 2N^2\int_0^{\infty} e^{-2x/a}dx = N^2 a\, e^{-2x/a}\Big|_0^{\infty} = N^2 a,$$

$$N = 1/\sqrt{a},$$

$$P(x>a) = N^2\int_a^{\infty} e^{-2|x|/a}dx = -\tfrac{1}{2}N^2 a\, e^{-2x/a}\Big|_a^{\infty} = \tfrac{1}{2}e^{-2} = 6.767\%.$$

(c) [3] $\psi(x) = \begin{cases} Nx^2(x-2a) & \text{for } 0<x<2a, \\ 0 & \text{otherwise.} \end{cases}$

$$1 = N^2\int_0^{2a}\left[x^2(x-2a)\right]^2 dx = N^2\int_0^{2a}\left(x^6 - 4ax^5 + 4a^2x^4\right)dx = N^2\left(\tfrac{1}{7}x^7 - \tfrac{2}{3}ax^6 + \tfrac{4}{5}a^2x^5\right)\Big|_0^{2a}$$

$$= \tfrac{128}{105}N^2a^7,$$

$$N = \sqrt{\tfrac{105}{128}}a^{-7/2}.$$

$$P(x>a) = N^2\int_a^{2a}\left[x^2(x-2a)\right]^2 dx = N^2\left(\tfrac{1}{7}x^7 - \tfrac{2}{3}ax^6 + \tfrac{4}{5}a^2x^5\right)\Big|_a^{2a} = \tfrac{105}{128}a^{-7}\left(\tfrac{128}{105}a^7 - \tfrac{29}{105}a^7\right) = \tfrac{99}{128}$$

$$= 77.34\%$$