Solutions to Chapter 5

5.3 [10] A particle of mass *m* is in a one-dimensional harmonic oscillator with angular frequency ω. If the particle is in the coherent state $|z\rangle$, find the uncertainties ΔX, ΔP, and check that they satisfy the uncertainty relation $\Delta X \Delta P \geq \frac{1}{2}\hbar$.

Our strategy will be to rewrite all operators in terms of the raising and lowering operators. Then, whenever we see *a* on the right, we'll rewrite it using $a|z\rangle = z|z\rangle$; whenever we see $a^\dagger$ on the left, we'll rewrite it using $\langle z|a^\dagger = \langle z|z^*$, and whenever we encounter $aa^\dagger$ we'll rewrite it as $aa^\dagger = a^\dagger a + 1$.

$$
\begin{aligned}
(\Delta X)^2 = \langle X^2\rangle - \langle X\rangle^2 &= \left(\sqrt{\frac{\hbar}{2m\omega}}\right)^2 \langle z|\left(a+a^\dagger\right)^2|z\rangle - \left(\sqrt{\frac{\hbar}{2m\omega}}\right)^2 \langle z|\left(a+a^\dagger\right)|z\rangle^2 \\
&= \frac{\hbar}{2m\omega}\left\{\langle z|\left(a^2 + aa^\dagger + a^\dagger a + a^{\dagger 2}\right)|z\rangle - \left[\langle z|\left(z+z^*\right)|z\rangle\right]^2\right\} \\
&= \frac{\hbar}{2m\omega}\left\{\langle z|\left(a^2 + 2a^\dagger a + 1 + a^{\dagger 2}\right)|z\rangle - \left(z+z^*\right)^2\right\} \\
&= \frac{\hbar}{2m\omega}\left\{\langle z|\left(z^2 + 2z^* z + 1 + z^{*2}\right)|z\rangle - \left(z+z^*\right)^2\right\} \\
&= \frac{\hbar}{2m\omega}\left(z^2 + 2z^* z + 1 + z^{*2} - z^2 - 2z^* z - z^{*2}\right) = \frac{\hbar}{2m\omega},
\end{aligned}
$$

$$
\begin{aligned}
(\Delta P)^2 = \langle P^2\rangle - \langle P\rangle^2 &= \left(i\sqrt{\hbar m\omega/2}\right)^2 \langle z|\left(a^\dagger - a\right)^2|z\rangle - \left(i\sqrt{\hbar m\omega/2}\right)^2 \langle z|\left(a^\dagger - a\right)|z\rangle^2 \\
&= -\tfrac{1}{2}\hbar m\omega\left\{\langle z|\left(a^{\dagger 2} - aa^\dagger - a^\dagger a + a^2\right)|z\rangle - \left[\langle z|\left(z^* - z\right)|z\rangle\right]^2\right\} \\
&= -\tfrac{1}{2}\hbar m\omega\left\{\langle z|\left(a^{\dagger 2} - 2a^\dagger a - 1 + a^2\right)|z\rangle - \left(z^* - z\right)^2\right\} \\
&= -\tfrac{1}{2}\hbar m\omega\left\{\langle z|\left(z^{*2} - 2z^* z - 1 + z^2\right)|z\rangle - \left(z^* - z\right)^2\right\} \\
&= -\tfrac{1}{2}\hbar m\omega\left(z^{*2} - 2z^* z - 1 + z^2 - z^{*2} + 2zz^* - z^2\right) = \tfrac{1}{2}\hbar m\omega .
\end{aligned}
$$

To summarize, taking the square root we have

$$
\Delta X = \sqrt{\frac{\hbar}{2m\omega}}, \quad \Delta P = \sqrt{\tfrac{1}{2}\hbar m\omega}, \quad (\Delta X)(\Delta P) = \tfrac{1}{2}\hbar
$$

The state satisfies the inequality by saturating it; that is, making it an equality. These states are commonly called *minimum uncertainty states* for this reason.

5.4 [15] In class we assumed that the coupled harmonic oscillators all had the same mass. Consider now the case where the oscillators have different masses, so that $H = \sum_i P_i^2 / 2m_i + \frac{1}{2} \sum_i \sum_j k_{ij} X_i X_j$ **.**

(a) [5] Rescale the variables *P_i* and *X_i* to new variables $\hat{P}_i$ and $\hat{X}_i$ with the usual commutation relations: $\left[\hat{X}_i, \hat{P}_j \right] = i\hbar\delta_{ij}$ **such that in terms of the new variables, the Hamiltonian is given by** $H = \sum_i \hat{P}_i^2 / 2m + \frac{1}{2} \sum_i \sum_j k_{ij}' \hat{X}_i \hat{X}_j$ **, where *m* is an arbitrary mass that you can choose.**

We want to rescale the momentum operators so that they all have the same coefficient in the kinetic energy term. If we define $P_i = c\hat{P}_i$ for some constant *c*, then the term that looked like $P_i^2 / 2m_i$ will become $c^2 \hat{P}_i^2 / 2m_i$, which we want to equal $\hat{P}_i^2 / 2m$, so $c^2 / 2m_i = 1/2m$, and we have $c = \sqrt{m_i / m}$, so that $\hat{P}_i = P_i / c = P_i \sqrt{m / m_i}$. To make sure the commutation relations are not messed up, we take the opposite factor for the *X*'s, so we have

$$\hat{P}_i = P_i \sqrt{m / m_i} \quad \text{and} \quad \hat{X}_i = X_i \sqrt{m_i / m}$$

Then these new variables will have commutation relations

$$\left[\hat{X}_i, \hat{P}_j \right] = \sqrt{m_i / m} \sqrt{m / m_j} \left[X_i, P_j \right] = i\hbar\delta_{ij} \sqrt{m_i / m_j} = i\hbar\delta_{ij} ,$$

where, at the final step, we have taken advantage of the fact that the expression is zero for $i \neq j$.

Solving for our old variables in terms of the new, we now work out the Hamiltonian in the new variables.

$$H = \sum_i \frac{1}{2m_i} P_i^2 + \frac{1}{2} \sum_i \sum_j k_{ij} X_i X_j = \sum_i \frac{1}{2m_i} \hat{P}_i^2 \left(\sqrt{m_i / m} \right)^2 + \frac{1}{2} \sum_i \sum_j k_{ij} \hat{X}_i \hat{X}_j \sqrt{m / m_i} \sqrt{m / m_j}$$

$$= \frac{1}{2m} \sum_i \hat{P}_i^2 + \frac{1}{2} \sum_i \sum_j \left(k_{ij} m / \sqrt{m_i m_j} \right) \hat{X}_i \hat{X}_j .$$

(b) [5] Find an expression for k_{ij}' in terms of the original variables *k_{ij}* and *m_i*, and explain in words how to obtain the eigenvalues of the original Hamiltonian.

Comparing the form we have for the Hamiltonian with the form requested, we see that $k_{ij}' = k_{ij}\, m / \sqrt{m_i m_j}$. The Hamiltonian is now in the same form as found in class. We now treat the constants k_{ij}' as a matrix k'. If we find the eigenvalues of k', which we will call k_i', then the normal modes of the harmonic oscillator will have frequencies $\omega_i = \sqrt{k_i' / m}$, and then the energy eigenstates will have energy $E_{n_1 \cdots n_N} = \sum_i \hbar\omega_i \left(n_i + \frac{1}{2} \right)$.

(c) [5] A system of two particles in one dimension has Hamiltonian $H = \dfrac{P_1^2}{2m} + \dfrac{P_2^2}{2(m/4)} + \frac{1}{2} m\omega^2 \left(5X_1^2 + 2X_2^2 + 2X_1X_2\right)$. Find the eigenvalues E_{ij} of this Hamiltonian.

We will simply use all the formulas we have already derived. The initial coupling has spring constants given by

$$k_{11} = 5m\omega^2, \quad k_{12} = k_{21} = m\omega^2, \quad k_{22} = 2m\omega^2$$

Note that we have split the cross-term in half. Now, the masses are $m_1 = m$ and $m_2 = m/4$. It is therefore straightforward to get the components of the k' matrix:

$$k'_{11} = \frac{mk_{11}}{\sqrt{mm}} = 5m\omega^2, \quad k'_{12} = k'_{21} = \frac{mk_{12}}{\sqrt{m\,m/4}} = 2m\omega^2, \quad \text{and} \quad k'_{22} = \frac{mk_{22}}{\sqrt{(m/4)^2}} = 8m\omega^2 .$$

We put this together into a single matrix

$$k' = m\omega^2 \begin{pmatrix} 5 & 2 \\ 2 & 8 \end{pmatrix}.$$

We need to find the eigenvalues of this matrix. Pulling out the common factors, we need to find the solutions of

$$0 = \det\begin{pmatrix} 5-\lambda & 2 \\ 2 & 8-\lambda \end{pmatrix} = \lambda^2 - 13\lambda + 40 - 4 = (\lambda - 9)(\lambda - 4).$$

Reintroducing the common factor, the eigenvalues of k' are $4m\omega^2$ and $9m\omega^2$.

We then find the frequencies $\omega_i = \sqrt{k'_i/m}$, which yields 2ω and 3ω. Thus the energy eigenvalues are

$$E_{ij} = \hbar(2\omega)\left(i + \tfrac{1}{2}\right) + \hbar(3\omega)\left(j + \tfrac{1}{2}\right) = \hbar\omega\left(2i + 3j + \tfrac{5}{2}\right).$$