Physics 741 – Graduate Quantum Mechanics 1 Solutions to Chapter 8

- [10] Suppose that L and S are two sets of angular momentum-like operators that commute with each other, so that [L_i, S_j] = 0. In this problem, you may assume from the commutation relations that it follows that [L², L] = 0 = [S², S].
 - (a) [2] Define J = L + S. Show that J is also an angular momentum-like operator. It follows automatically that $[J^2, J] = 0$.

We simply have to work out the commutator of each component of J with each other. We'll take advantage of the Levi-Civita tensor to show that

$$\begin{bmatrix} J_i, J_j \end{bmatrix} = \begin{bmatrix} L_i + S_i, L_j + S_j \end{bmatrix} = \begin{bmatrix} L_i, L_j \end{bmatrix} + \begin{bmatrix} S_i, S_j \end{bmatrix} = i\hbar \sum_k \varepsilon_{ijk} L_k + i\hbar \sum_k \varepsilon_{ijk} S_k = i\hbar \sum_k \varepsilon_{ijk} J_k$$

This saved us the work of doing it three times.

(b) [2] Show that $[L^2, J] = 0 = [S^2, J]$.

This is really pretty trivial.

$$\begin{bmatrix} \mathbf{L}^2, \mathbf{J} \end{bmatrix} = \begin{bmatrix} \mathbf{L}^2, \mathbf{L} \end{bmatrix} + \begin{bmatrix} \mathbf{L}^2, \mathbf{S} \end{bmatrix} = \mathbf{0} + \mathbf{0} = \mathbf{0}, \quad \begin{bmatrix} \mathbf{S}^2, \mathbf{J} \end{bmatrix} = \begin{bmatrix} \mathbf{S}^2, \mathbf{L} \end{bmatrix} + \begin{bmatrix} \mathbf{S}^2, \mathbf{S} \end{bmatrix} = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

(c) [3] Convince yourself (and me) that the four operators J^2 , L^2 , S^2 , and J_z all commute with each other (this is six commutators in all).

Since \mathbf{L}^2 and \mathbf{S}^2 commute with \mathbf{J} , it follows automatically that they commute with \mathbf{J}^2 and J_z (that's four commutators so far). Since all the **L**'s and **S**'s commute with each other, it follows that \mathbf{L}^2 and \mathbf{S}^2 commute with each other. Finally, as mentioned in part (a), since $[\mathbf{J}^2, \mathbf{J}] = 0$, \mathbf{J}^2 commutes with J_z .

(d) [3] Convince yourself that L_z and S_z do <u>not</u> commute with J^2 .

We simply try to do the commutation relations and see if it works.

$$\begin{bmatrix} \mathbf{J}^{2}, L_{z} \end{bmatrix} = \begin{bmatrix} (\mathbf{L} + \mathbf{S})^{2}, L_{z} \end{bmatrix} = \begin{bmatrix} \mathbf{L}^{2} + 2\mathbf{L} \cdot \mathbf{S} + \mathbf{S}^{2}, L_{z} \end{bmatrix} = \begin{bmatrix} \mathbf{L}^{2}, L_{z} \end{bmatrix} + 2\begin{bmatrix} \mathbf{L} \cdot \mathbf{S}, L_{z} \end{bmatrix} + \begin{bmatrix} \mathbf{S}^{2}, L_{z} \end{bmatrix}$$
$$= 2\begin{bmatrix} L_{x}, L_{z} \end{bmatrix} S_{x} + 2\begin{bmatrix} L_{y}, L_{z} \end{bmatrix} S_{y} + 2\begin{bmatrix} L_{z}, L_{z} \end{bmatrix} S_{z} = -2i\hbar L_{y}S_{x} + 2i\hbar L_{x}S_{y},$$
$$\begin{bmatrix} \mathbf{J}^{2}, S_{z} \end{bmatrix} = \begin{bmatrix} (\mathbf{L} + \mathbf{S})^{2}, S_{z} \end{bmatrix} = \begin{bmatrix} \mathbf{L}^{2} + 2\mathbf{L} \cdot \mathbf{S} + \mathbf{S}^{2}, S_{z} \end{bmatrix} = \begin{bmatrix} \mathbf{L}^{2}, S_{z} \end{bmatrix} + 2\begin{bmatrix} \mathbf{L} \cdot \mathbf{S}, S_{z} \end{bmatrix} + \begin{bmatrix} \mathbf{S}^{2}, S_{z} \end{bmatrix}$$
$$= 2L_{x}\begin{bmatrix} S_{x}, S_{z} \end{bmatrix} + 2L_{y}\begin{bmatrix} S_{y}, S_{z} \end{bmatrix} + 2L_{z}\begin{bmatrix} S_{z}, S_{z} \end{bmatrix} = -2i\hbar L_{x}S_{y} + 2i\hbar L_{y}S_{x}.$$

Not surprisingly, the sum of these two expressions is zero, which follows from $[\mathbf{J}^2, \mathbf{J}] = 0$.