Physics 741 – Graduate Quantum Mechanics 1

Final Exam, Fall 2016

Each question is worth 25 points, with points for each part marked separately. Some possibly useful formulas can be found at the end of the exam.

1. The Σ^{*0} baryon is a particle made of three quarks, one each up (u), down (d), and strange (s). These quarks are each spin $\frac{1}{2}$, and they are believed to have no orbital angular momentum.
   (a) [10] Based only on the spins of the quarks, if we measured $\mathbf{S}_u^2$, the spin on just the up quark what would we get? What could we possibly get if we measured the sum of the up and down quarks, $\mathbf{S}_{ud}^2$, where $\mathbf{S}_{ud} = \mathbf{S}_u + \mathbf{S}_d$? What if we measured the total spin squared, $\mathbf{S}^2 = \left(\mathbf{S}_u + \mathbf{S}_d + \mathbf{S}_s\right)^2$?
   (b) [8] In fact, the Σ^{*0} is known to be spin $\frac{3}{2}$. In light of this, how, at all, would your answers change to part (a)?
   (c) [7] A particular Σ^{*0} has its spin around the *z*-axis measured, and it is found to have eigenvalue $+\frac{1}{2}\hbar$. If this same Σ^{*0} were to have the *z*-component of the strange quark measured S_{sz} or the combined *z*-component spin of the up and down quarks $S_{ud,z}$, what would be the possible outcomes?

2. An electron is in a region with scalar potential $U = -\frac{1}{2}c\left(x^2 + y^2\right)$ and vector potential $\mathbf{A} = \frac{1}{2}b\left(x^2 + y^2\right)\hat{\mathbf{z}}$
   (a) [5] Find the electric and magnetic field from these potentials
   (b) [8] Write the Hamiltonian explicitly. Show that it commutes with one of the momentum operators. Which one?
   (c) [12] Find the commutator of L_z and S_z with the Hamiltonian *H*. Show that only one of L_z, S_z, and $J_z = L_z + S_z$ commutes with *H*. To save time, you may use the fact that L_z commutes with both $\mathbf{P}^2$ and with $X^2 + Y^2$.

3. The one dimensional infinite square well with width *a* for spinless particles of mass *m* has eigenstates $\left|\psi_n\right\rangle$, which have energy $E_n = \pi^2 n^2 \hbar^2 / \left(2ma^2\right)$.
   (a) [3] What are the corresponding energies for a three-dimensional infinite square well of side *a*?
   (b) [8] Suppose we have 14 spin-0 identical non-interacting particles in this same 3D square well. Would these particles be bosons or fermions? In the ground state, which states would be occupied, and what would be the energy of the ground state?
   (c) [14] Suppose we have 14 spin-½ identical non-interacting particles in this same 3D square well. Would these particles be bosons or fermions? In the ground state, which states would be occupied, and what would be the energy of the ground state?

4. Two electrons are in one dimension, one each in the two orthonormal wave functions $\phi_1(x) = \sqrt{\lambda} e^{-\lambda|x|}$ and $\phi_2(x) = \sqrt{2\lambda^3}\, x e^{-\lambda|x|}$. However, these wave functions do not take into account the electron spin, nor do they take into account the fact that they are fermions.
   (a) [9] Assume the two particles are in one of the two spin states $|\chi_\pm\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle \pm |-+\rangle\right)$. In each case, write out explicitly the properly normalized space wave function $\psi(x_1, x_2)$.
   (b) [8] For each case, find the probability that the first particle is at a positive position, $x_1 > 0$.
   (c) [8] For each case, find the probability that both particles are at a positive position, $x_1, x_2 > 0$.

5. A particle is in the state $|\psi\rangle = \begin{pmatrix} \cos\left(\frac{1}{2}\theta\right) \\ \sin\left(\frac{1}{2}\theta\right) \end{pmatrix}$, but the angle θ is unknown, and has a $\frac{1}{3}$ chance of being in each of the values $\theta = 0, \frac{1}{2}\pi, \pi$.
   (a) [8] Find the state operator ρ.
   (b) [9] Find the expectation value of the operators S_x and S_z for this state operator.
   (c) [8] Show that if the Hamiltonian is $H = aS_x$, the state operator will be time independent.

6. This problem is to be worked entirely in the Heisenberg formulation of quantum mechanics. Consider a particle of mass *m* in the one-dimensional linear potential $V(X) = -aX$.
   (a) [9] What is the Hamiltonian? Find expressions for the derivatives of the position operator *X* and momentum operator *P*.
   (b) [9] Solve for the position and momentum operators at time *t* in terms of the operators at time 0.
   (c) [7] Show that there is a minimum uncertainty relation between the uncertainty of the initial position $\Delta x(0)$ and the position at time *t*, $\Delta x(t)$.

Possibly Useful Formulas

Spin-½: $\mathbf{S} = \frac{1}{2}\hbar\boldsymbol{\sigma}$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

EM Fields

$$\mathbf{B} = \nabla \times \mathbf{A}$$
$$\mathbf{E} = -\partial\mathbf{A}/\partial t - \nabla U$$

State Operator

$$\rho = \sum_i f_i |\psi_i\rangle\langle\psi_i|$$
$$i\hbar \frac{d\rho}{dt} = [H, \rho]$$

Generalized Uncertainty

$$(\Delta a)(\Delta b) \geq \tfrac{1}{2}\left|\langle i[A,B]\rangle\right|$$

Heisenberg Evolution

$$\frac{d}{dt}A = \frac{i}{\hbar}[H, A]$$

EM Hamiltonian

$$H = \frac{\boldsymbol{\pi}^2}{2m} - eU + \frac{ge}{2m}\mathbf{B}\cdot\mathbf{S}$$

Possibly Useful Integral: $\int_0^\infty x^n e^{-\alpha x} dx = \alpha^{-n-1} n!$