Physics 741 **Name** ____________________________

# Final Exam, Fall 2022

Each question is worth 25 points, with points for each part marked separately. Some possibly useful formulas can be found at the end of the test.

1. A new particle has just been discovered! It is exactly like an electron *except* that in addition to having a Coulomb attraction to the proton $V_c(r) = -k_e e^2/r$, there is an additional repulsive term $V_r(r) = +6\hbar^2/\mu r^2$.
   (a) [4] Given that the potential depends only on *r*, what can we say about the bound states in terms of their dependence on angle and radius? Be as explicit as possible about the dependence on the angular part.
   (b) [5] Write an ordinary differential equation for the radial wave function *R*(*r*).
   (c) [7] Assume the ground state of modified hydrogen has $l = 0$. Based on this, show that the radial wave function for the *ground* state of modified hydrogen will be *exactly* like the equation for regular hydrogen with a different *l* value. What would be the corresponding *l* value for regular hydrogen?
   (d) [5] Write the full wave function $\psi(r,\theta,\phi)$ for the ground state of modified hydrogen. Some hydrogen wave functions can be found on the equation sheet.
   (e) [4] For ordinary hydrogen, the energy is given by $E_n = -\frac{\alpha^2 \mu c^2}{2n^2}$. What is the energy of the ground state for modified hydrogen?

2. A single electron lies in a hydrogen atom in the state $|\psi\rangle = \frac{1}{\sqrt{2}}\left|2,0,0,-\tfrac{1}{2}\right\rangle + \frac{i}{\sqrt{2}}\left|2,1,+1,+\tfrac{1}{2}\right\rangle$, in standard notation $|n,l,m,m_s\rangle$, where *n* corresponds to the energy, *l* to the eigenvalue of $\mathbf{L}^2$, *m* to $L_z$ and $m_s$ to $S_z$. For each of the following operators, find the expectation values:
   (a) [5] $\langle L_z \rangle$ (b) [5] $\langle \mathbf{L}^2 \rangle$ (c) [5] $\langle J_z \rangle$ (d) [10] $\langle \mathbf{J}^2 \rangle$

3. An electron of mass *m* lies in a region with electric field $\mathbf{E} = m\omega_0^2 (x\hat{\mathbf{x}} + y\hat{\mathbf{y}})/e$ and magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$.
   (a) [7] Show that these can be obtained with a scalar potential $U = a(x^2 + y^2)$ and vector potential $\mathbf{A}(\mathbf{r}) = b(x\hat{\mathbf{y}} - y\hat{\mathbf{x}})$, and find the coefficients *a* and *b*.
   (b) [7] Write the Hamiltonian explicitly. It may be helpful to expand it out.
   (c) [6] Demonstrate that this Hamiltonian commutes with one of the three momentum operators P, one of the three angular momentum operators L, and one of the three spin operators S. Which one, in each case? Do they commute with each other?
   (d) [5] Call the three eigenvalues under the operators $\hbar k$, $\hbar m$, and $\hbar m_s$ respectively. Are there any restrictions on these eigenvalues?

4. A free particle has wave function at $t = 0$ given by $\psi(x, t=0) = Ne^{-Ax^2/2}$.
   (a) [14] Find the wave function at arbitrary time $t$. You may wish to simplify a bit.
   (b) [5] Show that at $t = 0$, the formula reduces to wave function given at $t = 0$.
   (c) [6] Find the probability density as a function of $x$ at all times $t$. For full credit, simplify your answer as much as possible and make sure the result is real.

5. Twelve (12) non-interacting identical particles with spin are all in a 2D harmonic oscillator with angular frequency $\omega$, so the states would be labeled something like $|n_x, n_y, \chi\rangle$.
   (a) [4] What is the energy of a single particle in an arbitrary state of this type?
   (b) [5] If the particles have spin 0, would they be bosons or fermions? Which state(s) would be occupied in the ground state? What would be the ground state energy?
   (c) [8] If the particles have spin $\frac{1}{2}$, would they be bosons or fermions? Which state(s) would be occupied in the ground state? What would be the ground state energy?
   (d) [8] If the particles have spin $\frac{3}{2}$, would they be bosons or fermions? Which state(s) would be occupied in the ground state? What would be the ground state energy?

6. A spin ½-particle is in one of two states, given by $\Psi_\pm = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 2i \pm 1 \end{pmatrix}$, with equal probability.
   (a) [8] Find the state operator $\rho$. As a check, find its trace.
   (b) [9] Find the expectation value of all three spin operators $S_i = \frac{1}{2}\hbar\sigma_i$, where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

   (c) [8] The Hamiltonian is given by $H = \omega(S_z - S_y)$. Prove or disprove that $d\rho/dt = 0$.

## Possibly Useful Formulas

**Free Propagator:**

$$\Psi(x,t) = \int dx_0 K(x,t;x_0,t_0)\Psi(x_0,t_0)$$

$$K(x,t;x_0,t_0) = \sqrt{\frac{m}{2\pi i\hbar(t-t_0)}}\exp\left[\frac{im(x-x_0)^2}{2\hbar(t-t_0)}\right]$$

**Hydrogen Wave Functions**

$$R_{10} = \frac{2e^{-r/a}}{a^{3/2}}, \quad R_{30} = \frac{2e^{-r/3a}}{3\sqrt{3a^3}}\left(1 - \frac{2r}{3a} + \frac{2r^2}{27a^2}\right)$$

$$R_{32} = \frac{2\sqrt{2}r^2 e^{-r/3a}}{81\sqrt{15a^7}}, \quad R_{43} = \frac{r^3 e^{-r/4a}}{768\sqrt{35a^9}}$$

$$R_{40} = \frac{e^{-r/4a}}{4\sqrt{a^3}}\left(1 - \frac{3r}{4a} + \frac{r^2}{8a^2} - \frac{r^3}{192a^3}\right)$$

**Radial Wave Equation**

$$ER = -\frac{\hbar^2}{2\mu}\frac{1}{r}\frac{d^2}{dr^2}(rR) + \frac{(l^2+l)\hbar^2}{2\mu r^2}R + VR$$

**EM Fields**

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\mathbf{E} = -\partial\mathbf{A}/\partial t - \nabla U$$

**EM Hamiltonian**

$$H = \frac{\boldsymbol{\pi}^2}{2m} - eU + \frac{ge}{2m}\mathbf{B}\cdot\mathbf{S}$$

**Spherical Harmonics**

$$Y_0^0 = \frac{1}{\sqrt{4\pi}}$$

$$Y_2^2 = \sqrt{\frac{15}{32\pi}}e^{2i\phi}\sin^2\theta$$

$$Y_3^3 = \frac{\sqrt{35}}{8\sqrt{\pi}}e^{3i\phi}\sin^3\theta$$

**State Operator**

$$\rho = \sum_i f_i |\psi_i\rangle\langle\psi_i|$$

$$\langle A\rangle = \mathrm{Tr}(\rho A)$$

$$i\hbar\frac{d}{dt}\rho = [H,\rho]$$

**Possibly Useful Integral:**

$$\int_{-\infty}^{\infty} e^{-\alpha y^2/2-\beta y}dy = \sqrt{2\pi/\alpha}\,e^{\beta^2/2\alpha}$$


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