Physics 742 – Graduate Quantum Mechanics 2

Solutions to Second Exam, Spring 2018

The points for each question are marked. Each question is worth 20 points. Some possibly useful formulas appear at the end of the test.

1. A particle of mass *m* is in one dimension in the potential $V(x)=\begin{cases}\frac{1}{2}m\omega_+^2x^2 & \text{for } x>0,\\ \frac{1}{2}m\omega_-^2x^2 & \text{for } x<0.\end{cases}$

Estimate the energy of the *n*'th eigenstate using the WKB approximation.

We first label the energy *E* and need to find the turning points, where $E=V(x)$. We solve this separately for $x>0$ and $x<0$. We have

$$\text{for } x>0: \quad E=\tfrac{1}{2}m\omega_+^2x^2, \quad x^2=2E/m\omega_+^2, \quad x_+=\sqrt{2E/m\omega_+^2},$$
$$\text{for } x<0: \quad E=\tfrac{1}{2}m\omega_-^2x^2, \quad x^2=2E/m\omega_-^2, \quad x_-=-\sqrt{2E/m\omega_+^2}.$$

We then substitute into our WKB formula, which says

$$\begin{aligned}
\pi\hbar\left(n+\tfrac{1}{2}\right) &= \int_{x_-}^{x_+}\sqrt{2m\left[E-V(x)\right]}dx=\int_{-\sqrt{2E/m\omega_-^2}}^{\sqrt{2E/m\omega_+^2}}\sqrt{2m\left[E-V(x)\right]}dx \\
&= \int_{-\sqrt{2E/m\omega_-^2}}^{0}\sqrt{2mE-m^2\omega_-^2x^2}\,dx+\int_0^{\sqrt{2E/m\omega_+^2}}\sqrt{2mE-m^2\omega_+^2x^2}\,dx \\
&= \int_0^{\sqrt{2E/m\omega_-^2}}\sqrt{2mE-m^2\omega_-^2x^2}\,dx+\int_0^{\sqrt{2E/m\omega_+^2}}\sqrt{2mE-m^2\omega_+^2x^2}\,dx \\
&= \frac{2mE}{2\sqrt{m^2\omega_-^2}}\sin^{-1}\left(\sqrt{\frac{2E}{m\omega_-^2}}\sqrt{\frac{m^2\omega_-^2}{2E}}\right)+\frac{1}{2}\sqrt{\frac{2mE}{m^2\omega_-^2}}\sqrt{2mE-m^2\omega_-^2\frac{2mE}{m\omega_-^2}} \\
&\quad +\frac{2mE}{2\sqrt{m^2\omega_+^2}}\sin^{-1}\left(\sqrt{\frac{2E}{m\omega_+^2}}\sqrt{\frac{m^2\omega_+^2}{2E}}\right)+\frac{1}{2}\sqrt{\frac{2mE}{m^2\omega_+^2}}\sqrt{2mE-m^2\omega_+^2\frac{2mE}{m\omega_+^2}} \\
&= \frac{E}{\omega_-}\sin^{-1}(1)+\sqrt{\frac{E}{2m\omega_-}}\sqrt{2mE-2mE}+\frac{E}{\omega_+}\sin^{-1}(1)+\sqrt{\frac{E}{2m\omega_+}}\sqrt{2mE-2mE} \\
&= \frac{\pi}{2}E\left(\frac{1}{\omega_-}+\frac{1}{\omega_+}\right)=\frac{\pi E(\omega_-+\omega_+)}{2\omega_-\omega_+}.
\end{aligned}$$

We now simply solve this for the energy *E*, which yields

$$E=\hbar\frac{2\omega_+\omega_-}{\omega_++\omega_-}\left(n+\tfrac{1}{2}\right).$$

2. A particle in the ground state of a three-dimensional spherical infinite square well of radius *a* has wave function $\psi(\mathbf{r}) = \frac{1}{r\sqrt{2\pi a}}\sin\left(\frac{\pi r}{a}\right)$ in the allowed region $r < a$. The radius of this potential well is now increased to 2*a*. What is the probability that the particle remains in the ground state if the radius increases from *a* to 2*a* (a) adiabatically, or (b) suddenly?

In the adiabatic approximation, the probability of going from the ground state to the ground state is one, so no calculation needed.

$$P(|0\rangle \to |0'\rangle) = 1$$

In the sudden approximation, the wave function is assumed to not change, so we have $P(|0\rangle \to |0'\rangle) = |\langle 0'|0\rangle|^2$. The final wave function will be the same as the initial wave function, except that *a* will be replaced by 2*a*, so our new ground state wave functions is

$$\psi'(\mathbf{r}) = \frac{1}{r\sqrt{4\pi a}}\sin\left(\frac{\pi r}{2a}\right)$$

We now simply calculate the overlap integral, working in 3D of course. Since the initial state vanishes outside *r* > *a*, we only integrate to this radius. So we have

$$\begin{aligned}\langle 0'|0\rangle &= \int \psi'^*(\mathbf{r})\psi(r)\,d^3\mathbf{r} = \int \frac{1}{r\sqrt{4\pi a}}\sin\left(\frac{\pi r}{2a}\right)\frac{1}{r\sqrt{2\pi a}}\sin\left(\frac{\pi r}{a}\right)d^3\mathbf{r} \\ &= \frac{1}{2\pi a\sqrt{2}}\int d\Omega\int_0^a \sin\left(\frac{\pi r}{2a}\right)\sin\left(\frac{\pi r}{a}\right)\frac{r^2 dr}{r^2} = \frac{4\pi}{2\pi a\sqrt{2}}\int_0^a \sin\left(\frac{\pi r}{2a}\right)\sin\left(\frac{\pi r}{a}\right)dr \\ &= \frac{\sqrt{2}}{a}\frac{(-1)^{0+1}a\cdot 1}{\pi\left[\left(0+\frac{1}{2}\right)^2 - 1^2\right]} = \frac{-\sqrt{2}}{\pi\left(-\frac{3}{4}\right)} = \frac{4\sqrt{2}}{3\pi}.\end{aligned}$$

Hence the probability is

$$P(|0\rangle \to |0'\rangle) = |\langle 0'|0\rangle|^2 = \frac{32}{9\pi^2} = 36.0\%.$$

3. **An electron of mass *m* at *t* = 0 is in the ground state $\left|\Psi(t=0)\right\rangle = \left|0,0,0\right\rangle$ of a three-dimensional harmonic oscillator with frequency ω. In an attempt to excite it to a higher energy state, a small perturbation $W = \gamma XYte^{-\lambda t}$ is turned on starting at *t* = 0 and left on. To leading order, what states $\left|n,p,q\right\rangle$ (other than the ground state) can be excited, and what is the probability of it ending in this/these states at $t=\infty$?**

We first need to calculate the matrix elements $W_{FI}(t)$. If we call the final state $\left|\phi_F\right\rangle = \left|n,p,q\right\rangle$, then we have

$$W_{FI}(t) = \left\langle \phi_F \right| W(t) \left|\phi_I\right\rangle = \gamma te^{-\lambda t}\left\langle n,p,q\right| XY \left|0,0,0\right\rangle$$
$$= \gamma te^{-\lambda t}\frac{\hbar}{2m\omega}\left\langle n,p,q\right|\left(a_x + a_x^\dagger\right)\left(a_y + a_y^\dagger\right)\left|0,0,0\right\rangle = \frac{\gamma\hbar te^{-\lambda t}}{2m\omega}\left\langle n,p,q\right|\left(a_x + a_x^\dagger\right)\left|0,1,0\right\rangle$$
$$= \frac{\gamma\hbar te^{-\lambda t}}{2m\omega}\left\langle n,p,q \middle| 1,1,0\right\rangle.$$

Hence the only state that can become populated is this $\left|1,1,0\right\rangle$ state. The frequency difference between these two states is

$$\omega_{110,00} = \frac{E_{110} - E_{000}}{\hbar} = \frac{\hbar\omega\left(1+1+0+\frac{3}{2}\right) - \hbar\omega\left(0+0+0+\frac{3}{2}\right)}{\hbar} = \frac{2\hbar\omega}{\hbar} = 2\omega.$$

Then the transition amplitude is

$$S_{110,000} = 0 + \frac{1}{i\hbar}\int_0^\infty dt\, W_{FI}(t)e^{i\omega_{FI}t} = \frac{1}{i\hbar}\int_0^\infty \frac{\gamma\hbar te^{-\lambda t}}{2m\omega}e^{2i\omega t}dt = \frac{\gamma}{2m\omega i}\int_0^\infty te^{-\lambda t + 2i\omega t}dt$$
$$= \frac{\gamma 1!}{2m\omega i\left(\lambda - 2i\omega\right)^2}.$$

The probability for this transition is the square of the magnitude of this expression, or

$$P\left(\left|0,0,0\right\rangle \to \left|1,1,0\right\rangle\right) = \left|\frac{\gamma}{2m\omega i\left(\lambda - 2i\omega\right)^2}\right|^2 = \frac{\gamma^2}{4m^2\omega^2\left(\lambda - 2i\omega\right)^2\left(\lambda + 2i\omega\right)^2}$$
$$= \frac{\gamma^2}{4m^2\omega^2\left(\lambda^2 + 4\omega^2\right)^2}.$$

4. A system consists of a superposition of a zero photon state and a one photon states, so that $\left|\Psi\right\rangle = \frac{1}{\sqrt{3}}\left(\left|0\right\rangle - i\sqrt{2}\left|1,\mathbf{q},\tau\right\rangle\right)$ where $\mathbf{q} = q\hat{\mathbf{z}}$, and $\boldsymbol{\varepsilon}_{q\tau} = \hat{\mathbf{x}}$. What are the expectation values of the electric and magnetic field $\left\langle \mathbf{E}(\mathbf{r})\right\rangle$ and $\left\langle \mathbf{B}(\mathbf{r})\right\rangle$?

We start by simply writing

$$\begin{aligned}\left\langle \mathbf{E}(\mathbf{r})\right\rangle &= \left\langle \Psi\right|\mathbf{E}(\mathbf{r})\left|\Psi\right\rangle \\ &= \sum_{\mathbf{k},\sigma}\sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\frac{i}{3}\left(\left\langle 0\right| + i\sqrt{2}\left\langle 1,\mathbf{q},\tau\right|\right)\left(a_{\mathbf{k}\sigma}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}} - a^\dagger_{\mathbf{k}\sigma}\boldsymbol{\varepsilon}^*_{\mathbf{k}\sigma}e^{-i\mathbf{k}\cdot\mathbf{r}}\right)\left(\left|0\right\rangle - i\sqrt{2}\left|1,\mathbf{q},\tau\right\rangle\right), \\ \left\langle \mathbf{B}(\mathbf{r})\right\rangle &= \left\langle \Psi\right|\mathbf{B}(\mathbf{r})\left|\Psi\right\rangle \\ &= \sum_{\mathbf{k},\sigma}\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_k}}\frac{i}{3}\left(\left\langle 0\right| + i\sqrt{2}\left\langle 1,\mathbf{q},\tau\right|\right)\mathbf{k}\times\left(a_{\mathbf{k}\sigma}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}} - a^\dagger_{\mathbf{k}\sigma}\boldsymbol{\varepsilon}^*_{\mathbf{k}\sigma}e^{-i\mathbf{k}\cdot\mathbf{r}}\right)\left(\left|0\right\rangle - i\sqrt{2}\left|1,\mathbf{q},\tau\right\rangle\right),\end{aligned}$$

Now, in the infinite sum, any term that does not have the proper wave number and polarization will either create a photon that does not exist on the left, or it will annihilate a photon that isn't there on the right. In either case the contribution vanishes, so the only terms that contribute are when $\mathbf{k} = \mathbf{q}$ and $\tau = \sigma$, and for this polarization, $\boldsymbol{\varepsilon}_{q\tau} = \hat{\mathbf{x}}$ and $\mathbf{q} = q\hat{\mathbf{z}}$. We therefore have

$$\begin{aligned}\left\langle \mathbf{E}(\mathbf{r})\right\rangle &= \sqrt{\frac{\hbar\omega_q}{2\varepsilon_0 V}}\frac{i}{3}\left(\left\langle 0\right| + i\sqrt{2}\left\langle 1,\mathbf{q},\tau\right|\right)\left(a_{\mathbf{q}\tau}\hat{\mathbf{x}}e^{iq\hat{\mathbf{z}}\cdot\mathbf{r}} - a^\dagger_{\mathbf{q}\tau}\hat{\mathbf{x}}e^{-iq\hat{\mathbf{z}}\cdot\mathbf{r}}\right)\left(\left|0\right\rangle - i\sqrt{2}\left|1,\mathbf{q},\tau\right\rangle\right) \\ &= \hat{\mathbf{x}}\sqrt{\frac{\hbar\omega_q}{2\varepsilon_0 V}}\frac{i}{3}\left(\left\langle 0\right| + i\sqrt{2}\left\langle 1,\mathbf{q},\tau\right|\right)\left[e^{iqz}\left(0 - i\sqrt{2}\left|0\right\rangle\right) - e^{-iqz}\left(\left|1,\mathbf{q},\tau\right\rangle - 2i\left|2,\mathbf{q},\tau\right\rangle\right)\right] \\ &= \hat{\mathbf{x}}\sqrt{\frac{\hbar\omega_q}{2\varepsilon_0 V}}\frac{i}{3}\left[-i\sqrt{2}e^{iqz} - i\sqrt{2}e^{-iqz}\right] = \frac{1}{3}\sqrt{\frac{\hbar\omega_q}{\varepsilon_0 V}}\left(e^{iqz} + e^{-iqz}\right)\hat{\mathbf{x}} \\ &= \frac{2}{3}\sqrt{\frac{\hbar cq}{\varepsilon_0 V}}\cos(qz)\hat{\mathbf{x}}, \\ \left\langle \mathbf{B}(\mathbf{r})\right\rangle &= \sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_q}}\frac{i}{3}\left(\left\langle 0\right| + i\sqrt{2}\left\langle 1,\mathbf{q},\tau\right|\right)\mathbf{q}\times\left(a_{\mathbf{q}\tau}\hat{\mathbf{x}}e^{iq\hat{\mathbf{z}}\cdot\mathbf{r}} - a^\dagger_{\mathbf{q}\tau}\hat{\mathbf{x}}e^{-iq\hat{\mathbf{z}}\cdot\mathbf{r}}\right)\left(\left|0\right\rangle - i\sqrt{2}\left|1,\mathbf{q},\tau\right\rangle\right) \\ &= \mathbf{q}\times\hat{\mathbf{x}}\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_q}}\frac{i}{3}\left(\left\langle 0\right| + i\sqrt{2}\left\langle 1,\mathbf{q},\tau\right|\right)\left[e^{iqz}\left(0 - i\sqrt{2}\left|0\right\rangle\right) - e^{-iqz}\left(\left|1,\mathbf{q},\tau\right\rangle - 2i\left|2,\mathbf{q},\tau\right\rangle\right)\right] \\ &= q\hat{\mathbf{z}}\times\hat{\mathbf{x}}\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_q}}\frac{i}{3}\left[-i\sqrt{2}e^{iqz} - i\sqrt{2}e^{-iqz}\right] = \frac{q}{3}\sqrt{\frac{\hbar}{\varepsilon_0 Vcq}}\left(e^{iqz} + e^{-iqz}\right)\hat{\mathbf{y}} \\ &= \frac{2}{3}\sqrt{\frac{\hbar q}{\varepsilon_0 Vc}}\cos(qz)\hat{\mathbf{y}}.\end{aligned}$$

5. An electron is in the $|3,1,1\rangle$ state of the 3D cubical infinite square well with allowed region $x,y,z \in [0,a]$. Show that via the dipole transition, it can only decay to one of the states $|1,1,1\rangle$ or $|2,1,1\rangle$, and calculate the corresponding rate.

The wave function and energy for the state $|n,p,q\rangle$ is given by

$$\psi_{npq}(\mathbf{r}) = \sqrt{\frac{8}{a^3}}\sin\left(\frac{\pi nx}{a}\right)\sin\left(\frac{\pi py}{a}\right)\sin\left(\frac{\pi qz}{a}\right), \quad E_{npq} = \frac{\pi^2\hbar^2\left(n^2+p^2+q^2\right)}{2ma^2}.$$

We now have to calculate $\mathbf{r}_{FI} = \langle n,1,1|\mathbf{R}|3,1,1\rangle$ for each of the two final states. This is three separate integrals, for $\mathbf{R}$ corresponding to X, Y, and Z. If it does not correspond to X, then the resulting integral will contain the integral $\int_0^a \sin(\pi nx/a)\sin(3\pi x/a)\,dx = \frac{1}{2}\delta_{n3}$, which vanishes for each of the two final states. Hence the only component we need to calculate is X. For X, we have

$$\langle n,1,1|X|3,1,1\rangle = \frac{8}{a^3}\int_0^a x\sin\left(\frac{\pi nx}{a}\right)\sin\left(\frac{3\pi x}{a}\right)dx\int_0^a \sin\left(\frac{\pi y}{a}\right)\sin\left(\frac{\pi y}{a}\right)dy\int_0^a \sin\left(\frac{\pi z}{a}\right)\sin\left(\frac{\pi z}{a}\right)dz$$

$$= \frac{8}{a^3}\cdot\frac{a}{2}\cdot\frac{a}{2}\cdot\frac{2a^2 3n\left[(-1)^{n+3}-1\right]}{\pi^2\left(n^2-3^2\right)^2} = \frac{12an\left[(-1)^{n+3}-1\right]}{\pi^2\left(9-n^2\right)^2}.$$

Now, the numerator vanishes if $n = 1$, so $\langle 1,1,1|\mathbf{R}|3,1,1\rangle = 0$, and the only case we need to consider is $|2,1,1\rangle$, which we can then simplify to yield

$$\langle 2,1,1|\mathbf{R}|3,1,1\rangle = \frac{12a2[-1-1]}{\pi^2(9-4)^2}\hat{\mathbf{x}} = -\frac{48}{25\pi^2}a\hat{\mathbf{x}}.$$

We will also need the transition frequency, which is

$$\omega_{IF} = \frac{E_I - E_F}{\hbar} = \frac{\pi^2\hbar^2}{2ma^2\hbar}\left(3^2+1^2+1^2-2^2-1^2-1^2\right) = \frac{5\pi^2\hbar}{2ma^2}.$$

We now just substitute into our general formula to give

$$\Gamma = \frac{4\alpha}{3c^2}\omega_{IF}^3\left|\mathbf{r}_{FI}\right|^2 = \frac{4\alpha}{3c^2}\left(\frac{5\pi^2\hbar}{2ma^2}\right)^3\left(-\frac{48}{25\pi^2}a\hat{\mathbf{x}}\right)^2 = \frac{2^2\alpha}{3c^2}\cdot\frac{5^3\pi^6\hbar^3}{2^3m^3a^6}\cdot\frac{2^8 3^2 a^2}{5^4\pi^4} = \frac{384\pi^2\alpha\hbar^3}{5m^3a^4c^2}.$$

Possibly Helpful Formulas:

1D H.O.:

$$X=\sqrt{\frac{\hbar}{2m\omega}}\left(a+a^\dagger\right)$$

$$a|n\rangle=\sqrt{n}\,|n-1\rangle$$

$$a^\dagger|n\rangle=\sqrt{n+1}\,|n+1\rangle$$

Electric and Magnetic Field Operators

$$\mathbf{E}(\mathbf{r})=\sum_{\mathbf{k},\sigma}\sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\,i\left(a_{\mathbf{k}\sigma}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}}-a_{\mathbf{k}\sigma}^\dagger\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}^* e^{-i\mathbf{k}\cdot\mathbf{r}}\right)$$

$$\mathbf{B}(\mathbf{r})=\sum_{\mathbf{k},\sigma}\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_k}}\,i\mathbf{k}\times\left(a_{\mathbf{k}\sigma}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}}-a_{\mathbf{k}\sigma}^\dagger\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}^* e^{-i\mathbf{k}\cdot\mathbf{r}}\right)$$

1D Infinite Square Well

$$\psi_n(x)=\sqrt{2/a}\,\sin(\pi n x/a)$$

$$E_n=\frac{\pi^2\hbar^2 n^2}{2ma^2}$$

Spontaneous Decay:

$$\Gamma=\frac{4\alpha}{3c^2}\,\omega_{IF}^3\left|\mathbf{r}_{FI}\right|^2$$

Time-dependent Perturbation Theory

$$S_{FI}=\delta_{FI}+(i\hbar)^{-1}\int_0^T dt\,W_{FI}(t)\,e^{i\omega_{FI}t}+\cdots$$

WKB energies: $\int_a^b\sqrt{2m\left[E-V(x)\right]}\,dx=\pi\hbar\left(n+\tfrac{1}{2}\right)$

Possibly Helpful Integrals: In integrals below, *m* and *n* are non-negative integers

$$\int_0^\infty x^n e^{-\alpha x}dx=n!\,\alpha^{-(n+1)},\quad \int_0^y\sqrt{a-bx^2}\,dx=\frac{a}{2\sqrt{b}}\sin^{-1}\left(y\sqrt{b/a}\right)+\frac{y}{2}\sqrt{a-by^2}$$

$$\int_0^a\sin(\pi n x/a)\sin(\pi m x/a)\,dx=\int_0^a\sin\left[\pi\left(n+\tfrac{1}{2}\right)x/a\right]\sin\left[\pi\left(m+\tfrac{1}{2}\right)x/a\right]dx=\tfrac{1}{2}a\delta_{nm}$$

$$\int_0^a\sin(\pi n x/a)\sin\left[\pi\left(m+\tfrac{1}{2}\right)x/a\right]dx=4(-1)^{m+n}\,an\Big/\left[\pi\left(4m^2+4m+1-4n^2\right)\right],$$

$$\int_0^a x\sin\left(\frac{\pi n x}{a}\right)\sin\left(\frac{\pi m x}{a}\right)dx=\begin{cases}a^2/4 & \text{if } n=m,\\ 2a^2nm\left[(-1)^{n+m}-1\right]\Big/\left[\pi^2\left(n^2-m^2\right)^2\right] & \text{if } n\neq m.\end{cases}$$