Physics 742 – Graduate Quantum Mechanics 2

Solutions to Second Exam, Spring 2019

The points for each question are marked. Each question is worth 20 points. Some possibly useful formulas appear at the end of the test.

1. A particle of mass *m* is in the potential $V(x) = \begin{cases} Bx & \text{if } x > 0, \\ -Ax & \text{if } x < 0. \end{cases}$

Using the WKB approximation, estimate the energy of the *n*'th eigenstate.

We must first find the turning points for energy *E*; that is, the two points where $V(x) = E$. It is obvious one will be positive and one will be negative, and we easily find the negative one is $a = -E/A$ and the positive one is $b = E/B$. We then use the zeroth-order WKB approximation, which has the quantization condition

$$\begin{aligned}
\pi\hbar\left(n+\tfrac{1}{2}\right) &= \int_a^b \sqrt{2m\left[E-V(x)\right]}dx = \sqrt{2m}\int_{-E/A}^{E/B}\sqrt{E-V(x)}dx \\
&= \sqrt{2m}\left(\int_{-E/A}^{0}\sqrt{E+Ax}dx + \int_0^{E/B}\sqrt{E-Bx}dx\right) \\
&= \frac{2\sqrt{2m}}{3A}\left(E+Ax\right)^{3/2}\Big|_{-E/A}^{0} + \frac{2\sqrt{2m}}{-3B}\left(E-Bx\right)^{3/2}\Big|_0^{E/B} \\
&= \frac{2\sqrt{2m}}{3A}\left(E^{3/2}-0\right) + \frac{2\sqrt{2m}}{-3B}\left(0-E^{3/2}\right) = \frac{2\sqrt{2m}}{3}\frac{A+B}{AB}E^{3/2}.
\end{aligned}$$

We now just solve this for the energy *E*.

$$E^{3/2} = \frac{3\pi\hbar\left(n+\frac{1}{2}\right)AB}{2\sqrt{2m}\left(A+B\right)},$$

$$E = \frac{1}{2}\left[\frac{3\pi\hbar\left(n+\frac{1}{2}\right)AB}{\sqrt{m}\left(A+B\right)}\right]^{2/3}.$$

2. A particle is initially in the ground state $|0\rangle$ of the 1D harmonic oscillator with frequency ω_0 at $t = -\infty$. It is then subjected to a time-dependent perturbation of the form $W = \lambda X^2 t e^{-\alpha t^2}$. To leading order in λ, which state(s) $|n\rangle$, with $n \neq 0$, could it be excited to, and what would be the corresponding probability at $t = \infty$?

To leading order, $P(0 \to n) = \left|S_{n0}\right|^2$, where $S_{n0} = \delta_{n0} + (i\hbar)^{-1}\int_{-\infty}^{\infty} W_{n0}(t)e^{i\omega_{n0}t}dt$. Because we are interested in states with $n \neq 0$, we ignore the first term. We therefore need to find the matrix elements from *W*, which are given by

$$\langle n|W|0\rangle = \lambda t e^{-\alpha t^2}\langle n|X^2|0\rangle = \frac{\hbar\lambda}{2m\omega_0} t e^{-\alpha t^2}\langle n|\left(a+a^\dagger\right)^2|0\rangle = \frac{\hbar\lambda}{2m\omega_0} t e^{-\alpha t^2}\langle n|\left(a+a^\dagger\right)|1\rangle$$
$$= \frac{\hbar\lambda}{2m\omega_0} t e^{-\alpha t^2}\langle n|\left(\sqrt{2}|2\rangle + |0\rangle\right) = \frac{\hbar\lambda}{2m\omega_0} t e^{-\alpha t^2}\left(\sqrt{2}\delta_{n2} + \delta_{n0}\right).$$

We are again uninterested in $n = 0$, so the only relevant final state has $n = 2$. The frequency difference between these two states is

$$\omega_{2,0} = \left(E_2 - E_0\right)/\hbar = \left[\hbar\omega_0\left(2+\tfrac{1}{2}\right) - \hbar\omega_0\left(0+\tfrac{1}{2}\right)\right]/\hbar = 2\omega_0 .$$

The scattering matrix is then

$$S_{2,0} = \frac{1}{i\hbar}\int_{-\infty}^{\infty} W_{2,0}(t) e^{i\omega_{2,0}t} dt = \frac{\lambda\sqrt{2}}{2im\omega_0}\int_{-\infty}^{\infty} t e^{-\alpha t^2} e^{2i\omega_0 t} dt = \frac{\lambda\sqrt{2}\,2i\omega_0}{2m\omega_0 i 2\alpha}\sqrt{\frac{\pi}{\alpha}}\exp\left[\frac{(2i\omega_0)^2}{4\alpha}\right] = \frac{\lambda\sqrt{\pi}}{m\alpha\sqrt{2\alpha}} e^{-\omega_0^2/\alpha}.$$

The probability is just the square of the magnitude of this, or

$$P\left(|0\rangle \to |2\rangle\right) = \left|S_{2,0}\right|^2 = \frac{\pi\lambda^2}{2m^2\alpha^3} e^{-2\omega_0^2/\alpha} .$$

3. A particle of mass *m* in one dimension feels the potential $V(x) = -A\delta(x)$. This system is initially in the bound state, with wave function $\psi(x) = Ne^{-\lambda|x|}$, where $\lambda = mA/\hbar^2$.

(a) What is the normalization constant *N*?

This is easily determined from the normalization condition; namely,

$$1 = \int_{-\infty}^{\infty}\left|\psi(x)\right|^2 dx = N^2\int_{-\infty}^{\infty} e^{-2\lambda|x|}dx = 2N^2\int_0^{\infty} e^{-2\lambda x}dx = \frac{2N^2}{2\lambda} = \frac{N^2}{\lambda},$$
$$N = \sqrt{\lambda}.$$

(b) The strength of the δ-function is now increased from *A* to 2*A*. What is the probability the particle remains bound if the change is (i) sudden or (ii) adiabatic?

The initial wave function is $\psi(x) = \sqrt{\lambda}e^{-\lambda|x|}$. Because $\lambda \propto A$, doubling A also doubles λ, so the final wave function is $\psi'(x) = \sqrt{2\lambda}e^{-2\lambda|x|}$. In the sudden approximation, the probability of it going to this state is given by

$$P(\psi \to \psi') = \left|\langle\psi'|\psi\rangle\right|^2 = \left|\int_{-\infty}^{\infty}\psi'^*(x)\psi(x)dx\right|^2 = \left|\lambda\sqrt{2}\int_{-\infty}^{\infty} e^{-2\lambda|x|}e^{-\lambda|x|}dx\right|^2$$
$$= \left|2\lambda\sqrt{2}\int_0^{\infty} e^{-3\lambda x}dx\right|^2 = \left|\frac{2\lambda\sqrt{2}}{3\lambda}\right|^2 = \frac{8}{9}.$$

In the adiabatic approximation, the ground state goes to the ground state. Since there is only one bound state, this must be the ground state, so we have

$$P(\psi \to \psi') = 1.$$

4. At $t = 0$, photons are in the state $\left|\Psi(0)\right\rangle = \frac{1}{\sqrt{2}}\left(\left|n-1,\mathbf{q},\tau\right\rangle + \left|n,\mathbf{q},\tau\right\rangle\right)$, where $\mathbf{q} = q\hat{\mathbf{z}}$ and $\boldsymbol{\varepsilon}_{\mathbf{q}\tau} = \hat{\mathbf{x}}$. What is $\left|\Psi(t)\right\rangle$? Find the expectation value of the electric field at all times.

The state $\left|n,\mathbf{q},\tau\right\rangle$ represents n photons with wave number $\mathbf{q}$, and these photons have frequency $\omega = cq$, and hence energy $E = \hbar\omega_q$ each, for a total energy of $E = n\hbar cq$. It follows that an eigenstate will have the time-dependence $e^{-iEt/\hbar} = e^{-in\omega t}$. Each of the two terms are simply eigenstates of this sort, so we have

$$\left|\Psi(0)\right\rangle = \tfrac{1}{\sqrt{2}}\left(\left|n-1,\mathbf{q},\tau\right\rangle e^{-i\omega(n-1)t} + \left|n,\mathbf{q},\tau\right\rangle e^{-i\omega nt}\right) = \tfrac{1}{\sqrt{2}} e^{-i\omega nt}\left(\left|n-1,\mathbf{q},\tau\right\rangle e^{i\omega t} + \left|n,\mathbf{q},\tau\right\rangle\right),$$

where $\omega = cq$.

Now we need to find the expectation value of the electric field, which would be

$$\left\langle\Psi(t)\right|\mathbf{E}(t)\left|\Psi(t)\right\rangle = \frac{1}{2}e^{i\omega nt}\left(\left\langle n-1,\mathbf{q},\tau\right|e^{-i\omega t} + \left\langle n,\mathbf{q},\tau\right|e^{i\omega nt}\right)\sum_{\mathbf{k},\sigma}\sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\,i\left(a_{\mathbf{k}\sigma}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}} - a^\dagger_{\mathbf{k}\sigma}\boldsymbol{\varepsilon}^*_{\mathbf{k}\sigma}e^{-i\mathbf{k}\cdot\mathbf{r}}\right)$$
$$\cdot e^{-i\omega nt}\left(\left|n-1,\mathbf{q},\tau\right\rangle e^{i\omega t} + \left|n,\mathbf{q},\tau\right\rangle\right)$$

Now, in the sum, if **k** doesn't match **q** or σ doesn't match τ, then we are either creating a photon that we don't want or eliminating one that isn't there, so these terms vanish. Hence the sum collapses to a single term, and we have

$$\left\langle\mathbf{E}(t)\right\rangle = \frac{1}{2}\left(\left\langle n-1,\mathbf{q},\tau\right|e^{-i\omega t} + \left\langle n,\mathbf{q},\tau\right|\right)\sqrt{\frac{\hbar\omega_q}{2\varepsilon_0 V}}\,i\left(a_{\mathbf{q}\tau}\boldsymbol{\varepsilon}_{\mathbf{q}\tau}e^{i\mathbf{q}\cdot\mathbf{r}} - a^\dagger_{\mathbf{q}\tau}\boldsymbol{\varepsilon}^*_{\mathbf{q}\tau}e^{-i\mathbf{q}\cdot\mathbf{r}}\right)$$
$$\cdot\left(\left|n-1,\mathbf{q},\tau\right\rangle e^{i\omega t} + \left|n,\mathbf{q},\tau\right\rangle\right)$$

We know the polarization vector, and the only terms that will contribute is when we either go up by one photon or down by one photon, so this simplifies to

$$\left\langle\mathbf{E}(t)\right\rangle = \frac{1}{2}\hat{\mathbf{x}}\sqrt{\frac{\hbar\omega_q}{2\varepsilon_0 V}}\,i\left(e^{-i\omega t}\left\langle n-1,\mathbf{q},\tau\right|a_{\mathbf{q}\tau}e^{i\mathbf{q}\cdot\mathbf{r}}\left|n,\mathbf{q},\tau\right\rangle - \left\langle n,\mathbf{q},\tau\right|a^\dagger_{\mathbf{q}\tau}e^{-i\mathbf{q}\cdot\mathbf{r}}\left|n-1,\mathbf{q},\tau\right\rangle e^{i\omega t}\right)$$
$$= \frac{1}{2}\hat{\mathbf{x}}\sqrt{\frac{\hbar n\omega_q}{2\varepsilon_0 V}}\,i\left(e^{i\mathbf{q}\cdot\mathbf{r}-i\omega t} - e^{i\omega t-i\mathbf{q}\cdot\mathbf{r}}\right) = \hat{\mathbf{x}}\sqrt{\frac{\hbar n\omega_q}{2\varepsilon_0 V}}\,i^2\sin(\mathbf{q}\cdot\mathbf{r} - \omega t) = -\hat{\mathbf{x}}\sqrt{\frac{\hbar ncq}{2\varepsilon_0 V}}\sin(qz - qct).$$

5. An electron of mass *m* is in the state $\left|\psi_{1,1,3}\right\rangle$ of the 3D cubical infinite square well with allowed region $0 < x, y, z < a$. Find all state(s) to which it can decay in the dipole approximation, and the corresponding rate.

The wave functions and energies for the 3D cubical infinite square well are given on the equation sheet. To find the decay rate, we need to find the dipole matrix element $\left\langle\psi_{npq}\middle|\mathbf{R}\middle|\psi_{113}\right\rangle$. Furthermore, we know that the state can only go down in energy, so at least one of the three components must decrease. Now, let's look at one of these three matrix elements, say *X*, for which we have

$$\left\langle\psi_{npq}\middle|X\middle|\psi_{113}\right\rangle = \frac{8}{a^3}\int_0^a x\sin\left(\frac{\pi nx}{a}\right)\sin\left(\frac{\pi x}{a}\right)dx\int_0^a \sin\left(\frac{\pi py}{a}\right)\sin\left(\frac{\pi y}{a}\right)dy\int_0^a \sin\left(\frac{\pi qz}{a}\right)\sin\left(\frac{3\pi z}{a}\right)dz$$

$$= \frac{8}{a^3}\frac{2a^2 n\left[(-1)^{n+1}-1\right]}{\pi^2\left(n^2-1\right)^2}\cdot\frac{a}{2}\delta_{p1}\cdot\frac{a}{2}\delta_{q3}.$$

This vanishes unless $p = 1$ and $q = 3$. But we need to *decrease* the energy, so we need to then have *n* decrease. But *n* has its minimum value, so you can't decrease it. Hence this expression never contributes to decay. Similarly, the operator *Y* won't work either. The only one we need calculate is *Z*, for which we have

$$\left\langle\psi_{npq}\middle|Z\middle|\psi_{113}\right\rangle = \frac{8}{a^3}\int_0^a \sin\left(\frac{\pi nx}{a}\right)\sin\left(\frac{\pi x}{a}\right)dx\int_0^a \sin\left(\frac{\pi py}{a}\right)\sin\left(\frac{\pi y}{a}\right)dy\int_0^a z\sin\left(\frac{\pi qz}{a}\right)\sin\left(\frac{3\pi z}{a}\right)dz$$

$$= \frac{8}{a^3}\cdot\frac{a}{2}\delta_{n1}\cdot\frac{a}{2}\delta_{p1}\cdot\frac{6a^2 q\left[(-1)^{q+3}-1\right]}{\pi^2\left(q^2-3^2\right)^2} = \delta_{n1}\delta_{p1}\frac{12qa\left[(-1)^{q+3}-1\right]}{\pi^2\left(9-q^2\right)^2}.$$

Again, we want to decrease the energy, which demands $q < 3$, which means $q = 1$ or $q = 2$. The numerator vanishes if $q = 1$, so we have only one case that is non-zero, and we have

$$\left\langle\psi_{112}\middle|\mathbf{R}\middle|\psi_{113}\right\rangle = \frac{24a[-1-1]}{\pi^2(9-4)^2}\hat{\mathbf{z}} = -\frac{48a}{25\pi^2}\hat{\mathbf{z}}.$$

To complete the computation, we also need the frequency difference, which is

$$\omega_{IF} = \frac{1}{\hbar}\left(E_{113}-E_{112}\right) = \frac{1}{\hbar}\cdot\frac{\pi^2\hbar^2}{2ma^2}\left(1^2+1^2+3^2-1^2-1^2-2^2\right) = \frac{5\pi^2\hbar}{2ma^2}$$

We then substitute into the general formula for the decay rate, which yields

$$\Gamma = \frac{4\alpha}{3c^2}\omega_{IF}^3\left|\mathbf{r}_{FI}\right|^2 = \frac{4\alpha}{3c^2}\left(\frac{5\pi^2\hbar}{2ma^2}\right)^3\left(-\frac{48a}{25\pi^2}\right)^2 = \frac{384\pi^2\alpha\hbar^3}{5m^3c^2a^4}.$$

Possibly Helpful Formulas:

1D H.O.:

$$V(x) = \tfrac{1}{2} m\omega^2 x^2$$

$$X = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right)$$

$$a|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

3D Infinite Square Well

$$\psi_{npq} = \sqrt{\frac{8}{a^3}}\sin\left(\frac{\pi n x}{a}\right)\sin\left(\frac{\pi p y}{a}\right)\sin\left(\frac{\pi q z}{a}\right),$$

$$E_{npq} = \frac{\pi^2\hbar^2}{2ma^2}\left(n^2 + p^2 + q^2\right)$$

Spontaneous Decay:

$$\Gamma = \frac{4\alpha}{3c^2}\omega_{IF}^3 \left|\mathbf{r}_{FI}\right|^2$$

Time-dependent Perturbation Theory

$$S_{FI} = \delta_{FI} + (i\hbar)^{-1}\int_0^T dt\, W_{FI}(t)\, e^{i\omega_{FI} t} + \cdots$$

Electric field operator

$$\mathbf{E}(\mathbf{r}) = \sum_{\mathbf{k},\sigma}\sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\, i\left(a_{\mathbf{k}\sigma}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma} e^{i\mathbf{k}\cdot\mathbf{r}} - a_{\mathbf{k}\sigma}^\dagger \boldsymbol{\varepsilon}_{\mathbf{k}\sigma}^* e^{-i\mathbf{k}\cdot\mathbf{r}}\right)$$

Possibly Helpful Integrals:

$$\int_0^\infty x^n e^{-\alpha x}dx = n!\,\alpha^{-(n+1)}, \quad \int_{-\infty}^\infty e^{-Ax^2+Bx}dx = \sqrt{\frac{\pi}{A}}e^{B^2/4A}, \quad \int_{-\infty}^\infty x e^{-Ax^2+Bx}dx = \frac{B}{2A}\sqrt{\frac{\pi}{A}}e^{B^2/4A},$$

$$\int_{-\infty}^\infty x^2 e^{-Ax^2+Bx}dx = \left(\frac{B^2}{4A^2} + \frac{1}{2A}\right)\sqrt{\frac{\pi}{A}}e^{B^2/4A}, \quad \int(\alpha x + \beta)^n dx = \frac{1}{\alpha(n+1)}(\alpha x + \beta)^{n+1} + C$$

$$\int_0^a \sin\left(\frac{\pi n x}{a}\right)\sin\left(\frac{\pi p x}{a}\right)dx = \tfrac{1}{2}a\delta_{np}, \quad \int_0^a x\sin\left(\frac{\pi n x}{a}\right)\sin\left(\frac{\pi p x}{a}\right)dx = \frac{2a^2 np\left[(-1)^{n+p} - 1\right]}{\pi^2\left(n^2 - p^2\right)^2},$$