PHY 742 Spring 2025 Solutions to Second Exam

Each question is worth 20 points. Some possibly useful formulas appear below or on the handout

1. **A particle of mass μ with momentum $\hbar k$ scatters from a potential of the form $V(\mathbf{r}) = V_0 e^{-Ar^2}$. Using the first Born approximation, find the differential and total cross-section. I recommend working in Cartesian coordinates.**

According to the first Born approximation, we need to first find the Fourier transform of the potential, which is

$$\int d^3\mathbf{r}\,V(\mathbf{r})e^{-i\mathbf{K}\cdot\mathbf{r}} = V_0\int d^3\mathbf{r}\,e^{-Ar^2}e^{-i\mathbf{K}\cdot\mathbf{r}} = V_0\int_{-\infty}^{\infty}e^{-Ax^2-iK_xx}dx\int_{-\infty}^{\infty}e^{-Ay^2-iK_yy}dy\int_{-\infty}^{\infty}e^{-Az^2-iK_zz}dz$$

$$= V_0\left(\sqrt{\frac{\pi}{A}}\right)^3 e^{(iK_x)^2/4A}e^{(iK_y)^2/4A}e^{(iK_z)^2/4A} = \frac{V_0\pi^{3/2}}{A^{3/2}}e^{-\mathbf{K}^2/4A}.$$

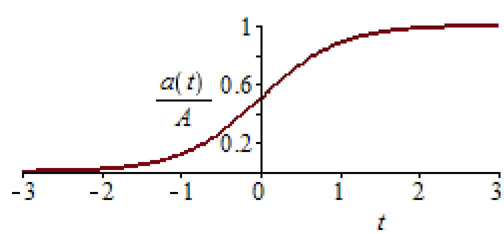
We now substitute this into the equation for the first Born approximation, which yields

$$\frac{d\sigma}{d\Omega} = \frac{\mu^2}{4\pi^2\hbar^4}\left(\frac{V_0\pi^{3/2}}{A^{3/2}}e^{-\mathbf{K}^2/4A}\right)^2 = \frac{\pi\mu^2V_0^2}{4\hbar^4A^3}e^{-\mathbf{K}^2/2A} = \frac{\pi\mu^2V_0^2}{4\hbar^4A^3}e^{-k^2(1-\cos\theta)/A}.$$

We then find the total cross-section

$$\sigma = \int\frac{d\sigma}{d\Omega}d\Omega = \frac{\pi\mu^2V_0^2}{4\hbar^4A^3}\int_{-1}^{1}e^{-k^2(1-\cos\theta)/A}d(\cos\theta)\int_0^{2\pi}d\phi = \frac{\pi^2\mu^2V_0^2}{2\hbar^4A^3}\left(-\frac{A}{k^2}\right)e^{-k^2(1-\cos\theta)/A}\Big|_{-1}^{1},$$

$$\sigma = \frac{\pi^2\mu^2V_0^2}{2\hbar^4A^2k^2}\left(1-e^{-2k^2/A}\right).$$



2. **A particle lies in a 1d moving harmonic oscillator with potential $V(x,t) = \frac{1}{2}m\omega^2\left[x - a(t)\right]^2$, where *a*(*t*) is a smoothly increasing function from $a(t=-\infty)=0$ to $a(t=\infty)=A$. It is initially in the ground state with wave function $\psi_0(x) = (m\omega/\pi\hbar)^{1/4}e^{-m\omega x^2/2\hbar}$. What is the probability that it is still in the ground state at $t=\infty$ if the increase is (a) adiabatic, (b) sudden?**

If the process is adiabatic, the probability is just 1. If it is sudden, we use the sudden approximation, where $P(|0\rangle \to |0'\rangle) = |\langle 0'|0\rangle|^2$. The final ground state will look just like the initial ground state, except it will be shifted to be centered at $x = A$, that is, $\psi_0'(x) = \psi_0(x-A)$. We therefore have

$$P\left(|0\rangle \to |0'\rangle\right) = \left|\langle 0'|0\rangle\right|^2 = \left|\sqrt{\frac{m\omega}{\pi\hbar}}\int_{-\infty}^{\infty} \exp\left(-\frac{m\omega x^2}{2\hbar}\right)\exp\left(-\frac{m\omega(x-A)^2}{2\hbar}\right)dx\right|^2$$

$$= \left|\sqrt{\frac{m\omega}{\pi\hbar}}\int_{-\infty}^{\infty} \exp\left(-\frac{m\omega}{2\hbar}\left(x^2+x^2-2xA+A^2\right)\right)dx\right|^2$$

$$= \left|\sqrt{\frac{m\omega}{\pi\hbar}}\exp\left(-\frac{m\omega}{2\hbar}A^2\right)\int_{-\infty}^{\infty} \exp\left(-\frac{m\omega}{\hbar}x^2+\frac{m\omega}{\hbar}xA\right)dx\right|^2$$

$$= \left|\sqrt{\frac{m\omega}{\pi\hbar}}\exp\left(-\frac{m\omega}{2\hbar}A^2\right)\sqrt{\frac{\pi\hbar}{m\omega}}\exp\left[\frac{\left(m\omega A/\hbar\right)^2}{4\left(m\omega/\hbar\right)}\right]\right|^2 = \left|\exp\left(-\frac{m\omega A^2}{4\hbar}\right)\right|^2,$$

$$P\left(|0\rangle \to |0'\rangle\right) = e^{-m\omega A^2/2\hbar}.$$

3. A spin-½ particle is in a magnetic field in the *z*-direction has two spin states $|\pm\rangle$ with energies $E_\pm = \mp\frac{1}{2}\hbar\omega_B$. It is initially in the spin up $|+\rangle$ state. It is subjected to a brief perturbation of the form $W(t) = \lambda S_x e^{-At^2}$. What is the probability that it flips to the $|-\rangle$ state? Recall that $S_x|\pm\rangle = \frac{1}{2}\hbar|\mp\rangle$.

To leading order, we use the formula $S_{FI} = \frac{1}{i\hbar}\int_{-\infty}^{\infty} dt\, W_{FI}(t)e^{i\omega_{FI}t}$, since the final state is different from the initial state. We need the frequency

$$\omega_{FI} = \frac{E_F - E_I}{\hbar} = \frac{E_- - E_+}{\hbar} = \frac{1}{\hbar}\left(\frac{1}{2}\hbar\omega_B - \left(-\frac{1}{2}\hbar\omega_B\right)\right) = \omega_B.$$

We now substitute this in to find the scattering matrix element, namely

$$S_{FI} = \frac{1}{i\hbar}\int_{-\infty}^{\infty} dt\, W_{FI}(t)e^{i\omega_{FI}t} = \frac{1}{i\hbar}\int_{-\infty}^{\infty} dt\, \langle -|\lambda S_x e^{-At^2}|+\rangle e^{i\omega_{FI}t}$$

$$= \frac{1}{i\hbar}\int_{-\infty}^{\infty} \lambda e^{-At^2+i\omega_B t}dt\, \langle -|S_x|+\rangle = \frac{\lambda}{i\hbar}\frac{1}{2}\hbar\int_{-\infty}^{\infty} e^{-At^2+i\omega_B t}dt = -\frac{\lambda i}{2}\sqrt{\frac{\pi}{A}}e^{(i\omega_B)^2/4A} = -\frac{\lambda i}{2}\sqrt{\frac{\pi}{A}}e^{-\omega_B^2/4A}.$$

The probability of a transition is then just

$$P(I \to F) = \left|S_{FI}\right|^2 = \frac{\pi\lambda^2}{4A}e^{-\omega_B^2/2A}.$$

4. **A system of pure photons is in the state $\left|\Psi\right\rangle = A\left(4\left|8,\mathbf{q},\tau\right\rangle - 3i\left|9,\mathbf{q},\tau\right\rangle\right)$, where $\mathbf{q} = q\hat{\mathbf{x}}$ and $\boldsymbol{\varepsilon}_{\mathbf{q}\tau} = \hat{\mathbf{y}}$. Find the normalization A and the expectation value of the magnetic field $\left\langle\Psi\right|\mathbf{B}(\mathbf{r})\left|\Psi\right\rangle$.**

To be normalized, we must have $\left\langle\Psi|\Psi\right\rangle = 1$, so

$$1 = \left\langle\Psi|\Psi\right\rangle = \left|\Psi\right\rangle = A^2\left(4\left\langle 8,\mathbf{q},\tau\right| + 3i\left\langle 9,\mathbf{q},\tau\right|\right)\left(4\left|8,\mathbf{q},\tau\right\rangle - 3i\left|9,\mathbf{q},\tau\right\rangle\right) = A^2\left(16+9\right) = 25A^2,$$

$$A = \frac{1}{5}.$$

The magnetic field operator has one raising and one lowering operator. The raising term will only be non-zero if it is increasing from 8 to nine photons, and only if the corresponding operator is $\mathbf{k} = \mathbf{q}$ and $\sigma = \tau$, so we have

$$\begin{aligned}
\left\langle\Psi\right|\mathbf{B}(\mathbf{r})\left|\Psi\right\rangle = \mathbf{B}(\mathbf{r}) &= \sum_{\mathbf{k},\sigma}\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_k}}\, i\mathbf{k}\times\left(a_{\mathbf{k}\sigma}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}} - a^\dagger_{\mathbf{k}\sigma}\boldsymbol{\varepsilon}^*_{\mathbf{k}\sigma}e^{-i\mathbf{k}\cdot\mathbf{r}}\right)\left|\Psi\right\rangle \\
&= \sum_{\mathbf{k},\sigma}\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_k}}\, i\mathbf{k}\times\left(\left\langle\Psi\right|a_{\mathbf{k}\sigma}\left|\Psi\right\rangle\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}} - \left\langle\Psi\right|a^\dagger_{\mathbf{k}\sigma}\left|\Psi\right\rangle\boldsymbol{\varepsilon}^*_{\mathbf{k}\sigma}e^{-i\mathbf{k}\cdot\mathbf{r}}\right) \\
&= \frac{i}{25}\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_{\mathbf{q}}}}\,\mathbf{q}\times\begin{bmatrix}4\left\langle 8,\mathbf{q},\tau\right|a_{\mathbf{q}\tau}\left|9,\mathbf{q},\tau\right\rangle(-3i)\boldsymbol{\varepsilon}_{\mathbf{q}\tau}e^{i\mathbf{q}\cdot\mathbf{r}} \\ -3i\left\langle 9,\mathbf{q},\tau\right|a^\dagger_{\mathbf{k}\sigma}\left|8,\mathbf{q},\tau\right\rangle 4\boldsymbol{\varepsilon}^*_{\mathbf{q}\tau}e^{-i\mathbf{q}\cdot\mathbf{r}}\end{bmatrix} \\
&= -\frac{12i^2 q}{25}\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_{\mathbf{q}}}}\, i\hat{\mathbf{x}}\times\left(\sqrt{9}\hat{\mathbf{y}}e^{iq\hat{\mathbf{x}}\cdot\mathbf{r}} + \sqrt{9}\hat{\mathbf{y}}e^{-iq\hat{\mathbf{x}}\cdot\mathbf{r}}\right) = \frac{36q}{25}\sqrt{\frac{\hbar}{2\varepsilon_0 Vcq}}\left(e^{iqx} + e^{-iqx}\right)\hat{\mathbf{z}} \\
&= \frac{72}{25}\sqrt{\frac{\hbar q}{2\varepsilon_0 Vc}}\cos(qx)\hat{\mathbf{z}}.
\end{aligned}$$

5. **An electron of mass *m* is in the $\left|3,2,1\right\rangle$ state of the 3D symmetric harmonic oscillator with angular frequency ω. In the dipole approximation, which states can it decay to, and what are the corresponding rates?**

We need to calculate the matrix elements $\left\langle n,p,q\right|\mathbf{R}\left|3,2,1\right\rangle$. We therefore start with

$$\begin{aligned}
\mathbf{R}\left|3,2,1\right\rangle = \left(\hat{\mathbf{x}}X + \hat{\mathbf{y}}Y + \hat{\mathbf{z}}Z\right)\left|3,2,1\right\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\left[\hat{\mathbf{x}}\left(a_x + a_x^\dagger\right) + \hat{\mathbf{y}}\left(a_y + a_y^\dagger\right) + \hat{\mathbf{z}}\left(a_z + a_z^\dagger\right)\right]\left|3,2,1\right\rangle \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left[\hat{\mathbf{x}}\left(\sqrt{3}\left|2,2,1\right\rangle + 2\left|4,2,1\right\rangle\right) + \hat{\mathbf{y}}\left(\sqrt{2}\left|3,1,1\right\rangle + \sqrt{3}\left|3,3,1\right\rangle\right) + \hat{\mathbf{z}}\left(\left|3,2,0\right\rangle + \sqrt{2}\left|3,2,2\right\rangle\right)\right]
\end{aligned}$$

We are only interested in cases where the energy goes down, so one of the numbers has to decrease. We have only three cases, which are

$$\langle 2,2,1|\mathbf{R}|3,2,1\rangle = \sqrt{\frac{3\hbar}{2m\omega}}\hat{\mathbf{x}}, \quad \langle 3,1,1|\mathbf{R}|3,2,1\rangle = \sqrt{\frac{\hbar}{m\omega}}\hat{\mathbf{y}}, \quad \langle 3,2,0|\mathbf{R}|3,2,1\rangle = \sqrt{\frac{\hbar}{2m\omega}}\hat{\mathbf{z}}.$$

We also need the energies to get the relevant frequencies. The energy of the state $|n,p,q\rangle$ is $\hbar\omega\left(n+p+q+\frac{3}{2}\right)$. The initial state has energy $E_I = \hbar\omega\left(6+\frac{3}{2}\right)$, and all of the final states have $E_F = \hbar\omega\left(5+\frac{3}{2}\right)$, so

$$\omega_{IF} = \frac{E_F - E_I}{\hbar} = \frac{\hbar\omega\left(6+\frac{3}{2}\right) - \hbar\omega\left(5+\frac{3}{2}\right)}{\hbar} = \omega.$$

The three rates are then calculated using $\Gamma = \frac{4\alpha}{3c^2}\omega_{IF}^3 \left|\mathbf{r}_{FI}\right|^2$ to yield

$$\Gamma\left(|3,2,1\rangle \to |2,2,1\rangle\right) = \frac{4\alpha}{3c^2}\omega^3\left(\sqrt{\frac{3\hbar}{2m\omega}}\right)^2 = \frac{2\alpha\hbar\omega^2}{mc^2},$$

$$\Gamma\left(|3,2,1\rangle \to |3,1,1\rangle\right) = \frac{4\alpha}{3c^2}\omega^3\left(\sqrt{\frac{\hbar}{m\omega}}\right)^2 = \frac{4\alpha\hbar\omega^2}{3mc^2},$$

$$\Gamma\left(|3,2,1\rangle \to |3,2,0\rangle\right) = \frac{4\alpha}{3c^2}\omega^3\left(\sqrt{\frac{\hbar}{2m\omega}}\right)^2 = \frac{2\alpha\hbar\omega^2}{3mc^2}.$$

Though it didn't ask for it, the branching ratios are $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{6}$ respectively.

Possibly Helpful Formulas:

Spontaneous Decay

$$\Gamma = \frac{4\alpha}{3c^2}\omega_{IF}^3\left|\mathbf{r}_{FI}\right|^2$$

1st Born Approximation

$$\frac{d\sigma}{d\Omega} = \frac{\mu^2}{4\pi^2\hbar^4}\left|\int d^3\mathbf{r}\,V(\mathbf{r})e^{-i\mathbf{K}\cdot\mathbf{r}}\right|^2$$

$$\mathbf{K}^2 = 2k^2\left(1-\cos\theta\right)$$

1D harmonic oscillator:

$$V(x) = \tfrac{1}{2}m\omega^2x^2$$

$$X = \sqrt{\frac{\hbar}{2m\omega}}\left(a+a^\dagger\right)$$

$$a|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

Magnetic Field Operator

$$\mathbf{B}(\mathbf{r}) = \sum_{\mathbf{k},\sigma}\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_k}}\,i\mathbf{k}\times\left(a_{\mathbf{k}\sigma}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}} - a_{\mathbf{k}\sigma}^\dagger\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}^*e^{-i\mathbf{k}\cdot\mathbf{r}}\right)$$

Time-dependent Pert. Theory

$$S_{FI} = \delta_{FI} + \frac{1}{i\hbar}\int_{-\infty}^{\infty}dt\,W_{FI}(t)\,e^{i\omega_{FI}t} + \cdots$$

Trigonometry: $e^{i\theta}+e^{-i\theta} = 2\cos\theta$, $\quad e^{i\theta}-e^{-i\theta} = 2i\sin\theta$.

Integrals: $\int_{-\infty}^{\infty}e^{-Ax^2+Bx}dx = \sqrt{\frac{\pi}{A}}e^{B^2/4A}$, $\quad \int_0^\infty x^n e^{-\alpha x}dx = n!\,\alpha^{-(n+1)}$.