Physics 742 – Graduate Quantum Mechanics 2

First Exam, Spring 2023

Please note that some possibly helpful formulas are listed below or on the handout. The value of each question is listed in square brackets at the start of the problem or part.

1. [10] A particle of mass *m* in one dimensionsis in the potential $V(x) = -\beta|x|^{-2/3}$. Using the WKB method, estimate the energy of the *n*'th eigenstate (the energies will be negative). I recommend making the substitution $x = y^{3/2}$ and then $z = \beta + Ey$.

2. [30] A particle of mass *m* in 3D is in the potential $V(x,y,z) = \frac{1}{2}m\omega^2(x^2+y^2+z^2) + \lambda xy$, where λ is small.
   (a) [4] Name and give the energies of the eigenstates of the unperturbed Hamiltonian in the limit $\lambda = 0$.
   (b) [13] For the ground state, find the state to first order in λ and the energy to second order in λ.
   (c) [13] For the first excited states, find the states to leading order and energies to first order in λ.

3. [20] A particle of mass *m* in 3D in the potential $V(r) = -\frac{2}{3}\beta r^{-3/2}$. Using the variational principle with trial wave function $\psi(r) = e^{-\lambda r/2}$, estimate the energy of the ground state.

4. [20] The hydrogen nucleus is normally assumed to be a point charge, leading to an electrostatic potential $U(r) = k_e e/r$. Suppose that the effect of the finite size of the nucleus is that this is modified to $U(r) = \frac{k_e e}{r}\left(1 - e^{-r/R}\right)$, where the scale of the nucleus *R* is much smaller than the radius of the atom *a*. Find an approximate formula for the energy shift ε', and argue that it vanishes except for s-wave state ($l = 0$). Find a formula for the shift in energy for the 1s state of hydrogen, with unperturbed wave function $\psi(r) = e^{-r/a}\big/\sqrt{\pi a^3}$.

5. [20] A particle of mass μ and wave number *k* scatters from a potential $V = \beta r^{-3/2}$, where β is small. Find the differential cross-section in the first Born approximation, and the total cross-section for scattering by angles $\theta > \frac{1}{2}\pi$. Hint: when doing the Fourier transform, I recommend doing the radial integral last.

Possibly Helpful Formulas:

Born Approximation

$$\frac{d\sigma}{d\Omega} = \frac{\mu^2}{4\pi^2\hbar^4}\left|\int d^3\mathbf{r}\,V(\mathbf{r})\,e^{-i\mathbf{K}\cdot\mathbf{r}}\right|^2$$

$$\mathbf{K}^2 = 2k^2(1-\cos\theta)$$

1D Harmonic Oscillator

$$X = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right)$$

$$a|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

Spherical Coordinates

$$\nabla\psi = \hat{\mathbf{r}}\frac{\partial\psi}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

Math

$$\sqrt{\pm i} = \frac{1}{\sqrt{\mp i}} = \frac{1}{\sqrt{2}}(1 \pm i)$$

Possibly Helpful Integrals

$$\int_0^\infty x^n e^{-Ax}\,dx = \begin{cases} n!/A^{n+1} & n \text{ integer,} \\ \Gamma(n+1)/A^{n+1} & \text{all } n. \end{cases}$$

$$\int_0^\pi e^{-iA\cos\theta}\sin\theta\,d\theta = 2\sin(A)/A,$$

$$\int_0^\infty x^n \sin(Bx)\,dx = \frac{\Gamma(n+1)}{B^{n+1}}\sin\left[\tfrac{1}{2}\pi(n+1)\right],$$

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}, \quad \Gamma\left(\tfrac{3}{2}\right) = \tfrac{1}{2}\sqrt{\pi}, \quad \Gamma\left(\tfrac{5}{2}\right) = \tfrac{3}{4}\sqrt{\pi}.$$

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