Physics 742 – Graduate Quantum Mechanics 2

Second Exam, Spring 2018

The points for each question are marked. Each question is worth 20 points. Some possibly useful formulas appear at the end of the test.

1. A particle of mass m is in one dimension in the potential $V(x)=\begin{cases}\frac{1}{2}m\omega_+^2x^2 & \text{for } x>0,\\ \frac{1}{2}m\omega_-^2x^2 & \text{for } x<0.\end{cases}$
   Estimate the energy of the n'th eigenstate using the WKB approximation.

2. A particle in the ground state of a three-dimensional spherical infinite square well of radius a has wave function $\psi(\mathbf{r})=\dfrac{1}{r\sqrt{2\pi a}}\sin\left(\dfrac{\pi r}{a}\right)$ in the allowed region $r<a$. The radius of this potential well is now increased to 2a. What is the probability that the particle remains in the ground state if the radius increases from a to 2a (a) adiabatically, or (b) suddenly?

3. An electron of mass m at t = 0 is in the ground state $|\Psi(t=0)\rangle=|0,0,0\rangle$ of a three-dimensional harmonic oscillator with frequency ω. In an attempt to excite it to a higher energy state, a small perturbation $W=\gamma XYte^{-\lambda t}$ is turned on starting at t = 0 and left on. To leading order, what states $|n,p,q\rangle$ (other than the ground state) can be excited, and what is the probability of it ending in this/these states at $t=\infty$?

4. A system consists of a superposition of a zero photon state and a one photon states, so that $|\Psi\rangle=\frac{1}{\sqrt{3}}\left(|0\rangle-i\sqrt{2}\,|1,\mathbf{q},\tau\rangle\right)$ where $\mathbf{q}=q\hat{\mathbf{z}}$, and $\boldsymbol{\varepsilon}_{q\tau}=\hat{\mathbf{x}}$. What are the expectation values of the electric and magnetic field $\langle\mathbf{E}(\mathbf{r})\rangle$ and $\langle\mathbf{B}(\mathbf{r})\rangle$?

5. An electron is in the $|3,1,1\rangle$ state of the 3D cubical infinite square well with allowed region $x,y,z\in[0,a]$. Show that via the dipole transition, it can only decay to one of the states $|1,1,1\rangle$ or $|2,1,1\rangle$, and calculate the corresponding rate.

Possibly Helpful Formulas:

1D H.O.:
$$X=\sqrt{\frac{\hbar}{2m\omega}}\left(a+a^\dagger\right)$$
$$a|n\rangle=\sqrt{n}\,|n-1\rangle$$
$$a^\dagger|n\rangle=\sqrt{n+1}\,|n+1\rangle$$

Electric and Magnetic Field Operators
$$\mathbf{E}(\mathbf{r})=\sum_{\mathbf{k},\sigma}\sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\,i\left(a_{\mathbf{k}\sigma}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}}-a^\dagger_{\mathbf{k}\sigma}\boldsymbol{\varepsilon}^*_{\mathbf{k}\sigma}e^{-i\mathbf{k}\cdot\mathbf{r}}\right)$$
$$\mathbf{B}(\mathbf{r})=\sum_{\mathbf{k},\sigma}\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_k}}\,i\mathbf{k}\times\left(a_{\mathbf{k}\sigma}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}}-a^\dagger_{\mathbf{k}\sigma}\boldsymbol{\varepsilon}^*_{\mathbf{k}\sigma}e^{-i\mathbf{k}\cdot\mathbf{r}}\right)$$

1D Infinite Square Well
$$\psi_n(x)=\sqrt{2/a}\,\sin(\pi nx/a)$$
$$E_n=\frac{\pi^2\hbar^2n^2}{2ma^2}$$

Spontaneous Decay:
$$\Gamma=\frac{4\alpha}{3c^2}\omega_{IF}^3\left|\mathbf{r}_{FI}\right|^2$$

Time-dependent Perturbation Theory
$$S_{FI}=\delta_{FI}+(i\hbar)^{-1}\int_0^T dt\,W_{FI}(t)e^{i\omega_{FI}t}+\cdots$$

WKB energies: $\int_a^b\sqrt{2m\left[E-V(x)\right]}\,dx=\pi\hbar\left(n+\frac{1}{2}\right)$

Possibly Helpful Integrals: In integrals below, *m* and *n* are non-negative integers

$$\int_0^\infty x^n e^{-\alpha x}dx = n!\alpha^{-(n+1)}, \quad \int_0^y \sqrt{a-bx^2}\,dx = \frac{a}{2\sqrt{b}}\sin^{-1}\left(y\sqrt{b/a}\right)+\frac{y}{2}\sqrt{a-by^2}$$

$$\int_0^a \sin(\pi nx/a)\sin(\pi mx/a)\,dx = \int_0^a \sin\left[\pi\left(n+\tfrac{1}{2}\right)x/a\right]\sin\left[\pi\left(m+\tfrac{1}{2}\right)x/a\right]dx = \tfrac{1}{2}a\delta_{nm}$$

$$\int_0^a \sin(\pi nx/a)\sin\left[\pi\left(m+\tfrac{1}{2}\right)x/a\right]dx = 4(-1)^{m+n}\,an\Big/\left[\pi\left(4m^2+4m+1-4n^2\right)\right],$$

$$\int_0^a x\sin\left(\frac{\pi nx}{a}\right)\sin\left(\frac{\pi mx}{a}\right)dx = \begin{cases} a^2/4 & \text{if } n=m, \\ 2a^2nm\left[(-1)^{n+m}-1\right]\Big/\left[\pi^2\left(n^2-m^2\right)^2\right] & \text{if } n\neq m. \end{cases}$$

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