Physics 742 – Graduate Quantum Mechanics 2

Second Exam, Spring 2019

The points for each question are marked. Each question is worth 20 points. Some possibly useful formulas appear at the end of the test.

1. A particle of mass *m* is in the potential $V(x)=\begin{cases} Bx & \text{if } x>0, \\ -Ax & \text{if } x<0. \end{cases}$

   Using the WKB approximation, estimate the energy of the *n*'th eigenstate.

2. A particle is initially in the ground state $|0\rangle$ of the 1D harmonic oscillator with frequency ω_0 at $t=-\infty$. It is then subjected to a time-dependent perturbation of the form $W=\lambda X^2 t e^{-\alpha t^2}$. To leading order in λ, which state(s) $|n\rangle$, with $n\neq 0$, could it be excited to, and what would be the corresponding probability at $t=\infty$?

3. A particle of mass *m* in one dimension feels the potential $V(x)=-A\delta(x)$. This system is initially in the bound state, with wave function $\psi(x)=Ne^{-\lambda|x|}$, where $\lambda=mA/\hbar^2$.
   (a) What is the normalization constant *N*?
   (b) The strength of the δ-function is now increased from *A* to 2*A*. What is the probability the particle remains bound if the change is (i) sudden or (ii) adiabatic?

4. At *t* = 0, photons are in the state $|\Psi(0)\rangle=\frac{1}{\sqrt{2}}\left(|n-1,\mathbf{q},\tau\rangle+|n,\mathbf{q},\tau\rangle\right)$, where $\mathbf{q}=q\hat{\mathbf{z}}$ and $\boldsymbol{\varepsilon}_{\mathbf{q}\tau}=\hat{\mathbf{x}}$. What is $|\Psi(t)\rangle$? Find the expectation value of the electric field at all times.

5. An electron of mass *m* is in the state $|\psi_{1,1,3}\rangle$ of the 3D cubical infinite square well with allowed region $0<x,y,z<a$. Find all state(s) to which it can decay in the dipole approximation, and the corresponding rate.

Possibly Helpful Formulas:

1D H.O.:

$$V(x)=\tfrac{1}{2}m\omega^2x^2$$

$$X=\sqrt{\frac{\hbar}{2m\omega}}\left(a+a^\dagger\right)$$

$$a|n\rangle=\sqrt{n}\,|n-1\rangle$$

$$a^\dagger|n\rangle=\sqrt{n+1}\,|n+1\rangle$$

3D Infinite Square Well

$$\psi_{npq}=\sqrt{\frac{8}{a^3}}\sin\left(\frac{\pi nx}{a}\right)\sin\left(\frac{\pi py}{a}\right)\sin\left(\frac{\pi qz}{a}\right),$$

$$E_{npq}=\frac{\pi^2\hbar^2}{2ma^2}\left(n^2+p^2+q^2\right)$$

Spontaneous Decay:

$$\Gamma=\frac{4\alpha}{3c^2}\omega_{IF}^3\left|\mathbf{r}_{FI}\right|^2$$

Time-dependent Perturbation Theory

$$S_{FI}=\delta_{FI}+(i\hbar)^{-1}\int_0^T dt\,W_{FI}(t)\,e^{i\omega_{FI}t}+\cdots$$

Electric field operator

$$\mathbf{E}(\mathbf{r})=\sum_{\mathbf{k},\sigma}\sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\,i\left(a_{\mathbf{k}\sigma}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}}-a_{\mathbf{k}\sigma}^\dagger\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}^*e^{-i\mathbf{k}\cdot\mathbf{r}}\right)$$

Possibly Helpful Integrals:

$$\int_0^\infty x^n e^{-\alpha x} dx = n!\alpha^{-(n+1)}, \quad \int_{-\infty}^\infty e^{-Ax^2+Bx} dx = \sqrt{\frac{\pi}{A}} e^{B^2/4A}, \quad \int_{-\infty}^\infty x e^{-Ax^2+Bx} dx = \frac{B}{2A}\sqrt{\frac{\pi}{A}} e^{B^2/4A},$$

$$\int_{-\infty}^\infty x^2 e^{-Ax^2+Bx} dx = \left(\frac{B^2}{4A^2}+\frac{1}{2A}\right)\sqrt{\frac{\pi}{A}} e^{B^2/4A}, \quad \int(\alpha x+\beta)^n\, dx = \frac{1}{\alpha(n+1)}(\alpha x+\beta)^{n+1}+C$$

$$\int_0^a \sin\left(\frac{\pi n x}{a}\right)\sin\left(\frac{\pi p x}{a}\right) dx = \tfrac{1}{2} a\delta_{np}, \quad \int_0^a x\sin\left(\frac{\pi n x}{a}\right)\sin\left(\frac{\pi p x}{a}\right) dx = \frac{2a^2 np\left[(-1)^{n+p}-1\right]}{\pi^2\left(n^2-p^2\right)^2}.$$

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