Physics 742 – Graduate Quantum Mechanics 2

Second Exam, Spring 2023

The points for each question are marked. Each question is worth 20 points. Some possibly useful formulas appear at the end of the test or on the handout.

1. A particle of mass m is in the ground state of a 3D harmonic oscillator with time-dependent frequency, so $V(\mathbf{r},t)=\frac{1}{2}m\omega^2(t)\mathbf{r}^2$. At $t=-\infty$, $\omega(t)=\omega_0$, while at $t=+\infty$, $\omega(t)=2\omega_0$. What is the probability it ends in the ground state if the process is (a) adiabatic, or (b) sudden?

2. A particle is in the ground state $|0,0,0\rangle$ of a 3D harmonic oscillator with frequency ω_0. A perturbation of the form $W=\alpha XYZ$ is turned on at time $t=0$. Find the probability to leading (second) order in α that it is in some other state at time T.

3. Suppose that we have a solution $\Psi(\mathbf{r},t)$ of the free Dirac equation, so $i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t)=\left(-i\hbar c\boldsymbol{\alpha}\cdot\nabla+mc^2\beta\right)\Psi(\mathbf{r},t)$. Show that $\Psi(-\mathbf{r},t)$ is generally *not* a solution, but that $\beta\Psi(-\mathbf{r},t)$ is, where β is the matrix appearing in the Dirac equation.

4. An electromagnetic system is in a superposition of zero or one photons state, $|\psi\rangle=N\left(\sqrt{2}|0\rangle+|1,\mathbf{q},1\rangle+i|1,\mathbf{q},2\rangle\right)$, where $\mathbf{q}=q\hat{\mathbf{z}}$, $\boldsymbol{\varepsilon}_{\mathbf{q}1}=\hat{\mathbf{x}}$ and $\boldsymbol{\varepsilon}_{\mathbf{q}2}=\hat{\mathbf{y}}$. Find the normalization N and the electric field expectation value $\langle\psi|\mathbf{E}(\mathbf{r})|\psi\rangle$. Your final answer should be manifestly real.

5. An electron of mass m is in a 3D cubical infinite square well of size a in the state $|n_x,n_y,n_z\rangle=|4,1,1\rangle$. Find the dipole matrix element $\mathbf{r}_{FI}$ between this and any of the lower energy states $|n,1,1\rangle$, and the rate Γ to decay to each of these states.

Possibly Helpful Formulas:

1D harmonic oscillator:

$$V(x)=\tfrac{1}{2}m\omega^2x^2$$

$$X=\sqrt{\frac{\hbar}{2m\omega}}\left(a+a^\dagger\right)$$

$$a|n\rangle=\sqrt{n}|n-1\rangle$$

$$a^\dagger|n\rangle=\sqrt{n+1}|n+1\rangle$$

$$\psi_0(x)=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}e^{-m\omega x^2/2\hbar}$$

Time-dependent Perturbation Theory

$$S_{FI}=\delta_{FI}+\frac{1}{i\hbar}\int_0^T dt\,W_{FI}(t)e^{i\omega_{FI}t}+\cdots$$

Trigonometry

$$\cos(\theta)=\tfrac{1}{2}\left(e^{i\theta}+e^{-i\theta}\right)$$

$$\sin(\theta)=\tfrac{1}{2i}\left(e^{i\theta}-e^{-i\theta}\right)$$

Dirac Matrices

$$\alpha_i\beta=-\beta\alpha_i$$

$$\alpha_i\alpha_j=-\alpha_j\alpha_i,\ i\neq j$$

$$\alpha_i^2=\beta^2=1$$

Spontaneous Decay

$$\Gamma=\frac{4\alpha}{3c^2}\omega_{IF}^3\left|\mathbf{r}_{FI}\right|^2$$

1D infinite square well:

$$\phi_n(x)=\sqrt{\frac{2}{a}}\sin\left(\frac{\pi nx}{a}\right)$$

$$E_n=\frac{\pi^2n^2\hbar^2}{2ma^2}$$

Electric field operator

$$\mathbf{E}(\mathbf{r})=\sum_{\mathbf{k},\sigma}\sqrt{\frac{\hbar\omega_k}{2\varepsilon_0V}}i\left(a_{\mathbf{k}\sigma}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}}-a_{\mathbf{k}\sigma}^\dagger\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}^*e^{-i\mathbf{k}\cdot\mathbf{r}}\right)$$

Possibly Helpful Integrals:

The formulas below assume *n* and *p* are non-negative integers

$$\int_0^\infty x^n e^{-Ax^2}dx = \frac{\Gamma\left(n+\frac{1}{2}\right)}{2A^{n+\frac{1}{2}}},\quad \Gamma\left(\tfrac{1}{2}\right)=\sqrt{\pi},\quad \Gamma(1)=1,\quad \Gamma\left(\tfrac{3}{2}\right)=\tfrac{1}{2}\sqrt{\pi},\quad \Gamma(2)=1,\quad \Gamma\left(\tfrac{5}{2}\right)=\tfrac{3}{4}\sqrt{\pi}.$$

$$\int_0^a \sin\left(\frac{\pi nx}{a}\right)\sin\left(\frac{\pi px}{a}\right)dx = \tfrac{1}{2}a\delta_{np}\ ,\quad \int_0^a x\sin\left(\frac{\pi nx}{a}\right)\sin\left(\frac{\pi px}{a}\right)dx = \frac{-2a^2pn\left[1-(-1)^{n+p}\right]}{\pi^2\left(n^2-p^2\right)^2}.$$

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