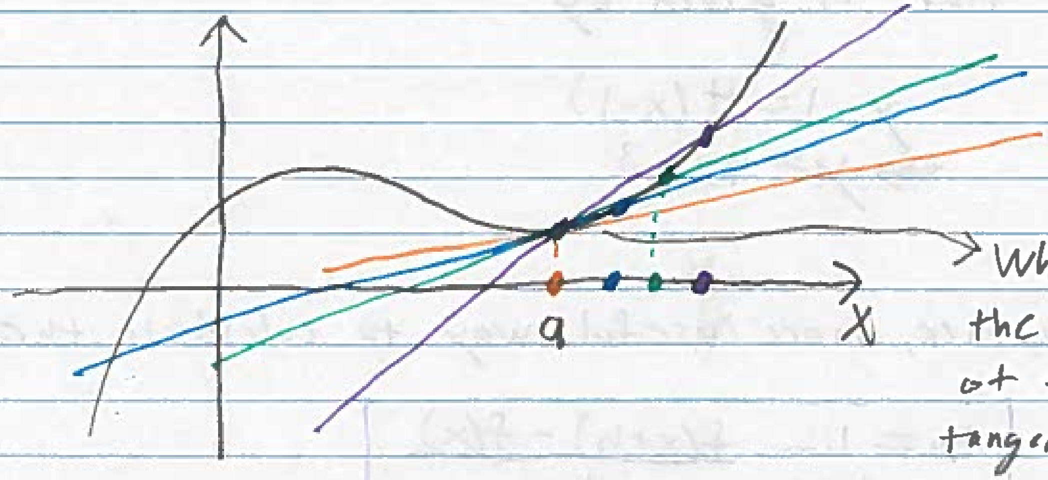
Lecture 6: Derivatives and Rates of Change



Definition - The slope of the tangent line at a is given by:

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \qquad \frac{(\text{Rise})}{(\text{Run})}$$

example:

If $f(x) = x^4$ find the equation of the tangent line at $x = 1$.

$$m = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$$

$$= \lim_{x \to 1} \frac{x^4 - 1}{x - 1}$$

$$= \lim_{x \to 1} \frac{(x^2 - 1)(x^2 + 1)}{x - 1}$$

$$= \lim_{x \to 1} \frac{\cancel{(x-1)}(x+1)(x^2+1)}{\cancel{x-1}}$$

$$= 4.$$

Therefore, since $f(1)=1$ the equation of the line is given by

$$y-1=4(x-1)$$
$$\Rightarrow y=4x-3.$$

Another, more useful way to calculate the slope is:

$$\boxed{m=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}},$$

example:

Find an equation of the tangent line to the function $f(x)=x^{-1/2}$ at $x=1$.

$$m=\lim_{h\to 0}\frac{\frac{1}{\sqrt{x+h}}-\frac{1}{\sqrt{x}}}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x}\sqrt{x+h}\cdot h}$$

$$=\lim_{h\to 0}\frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x}\cdot\sqrt{x+h}\cdot h}\cdot\frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}}$$

$$=\lim_{h\to 0}\frac{\cancel{x}-\cancel{x}-\cancel{h}}{\sqrt{x}\cdot\sqrt{x+h}\cdot\cancel{h}(\sqrt{x}+\sqrt{x+h})}$$

$$=\lim_{h\to 0}\frac{-1}{\sqrt{x}\sqrt{x+h}\cdot(\sqrt{x}+\sqrt{x+h})}$$

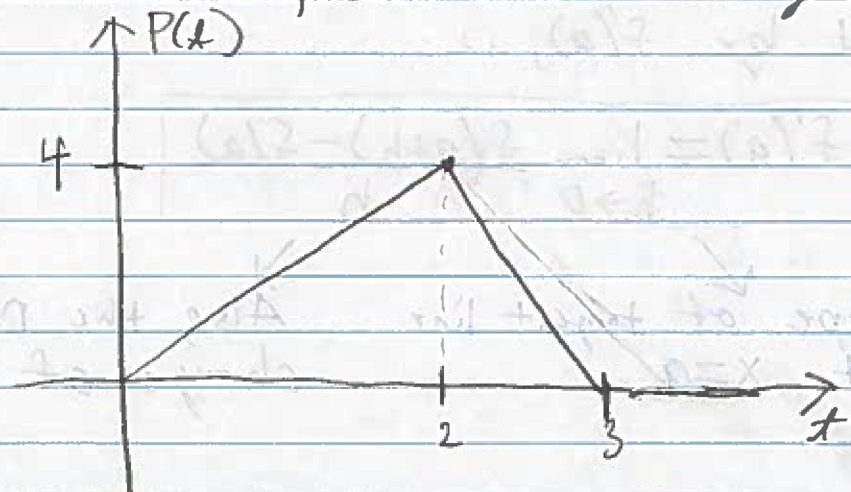
$$=-\frac{1}{2x^{3/2}}$$

Therefore at $x=1$, $m=-\frac{1}{2}$. The equation of the line is therefore

$$y=-\tfrac{1}{2}(x-1)-1$$

Velocity -

$P(t)$ is the position of something (car, ball, etc).



How fast is the car travelling?
When is the car moving forward? Backward?

$$\text{rate} \times \text{time} = \text{distance} \Rightarrow \frac{\text{distance}}{\text{time}} = \text{rate}$$

For $0 \le t \le 2$, $v = \frac{4}{2} = 2$
For $2 \le t \le 3$, $v = -\frac{4}{1} = -4$

For $P(t)$ the position, the velocity at time t is given by

$$\boxed{V(t) = \lim_{\Delta t \to 0} \frac{P(t+\Delta t) - P(t)}{\Delta t} = \frac{\text{Change in distance}}{\text{Change in time}}}$$

Derivative:

The derivative of a function f at a number a, denoted by $f'(a)$, is

$$\boxed{f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

slope of tangent line at $x=a$

Also, the rate of change of $f(x)$ at $x=a$