Homework 1

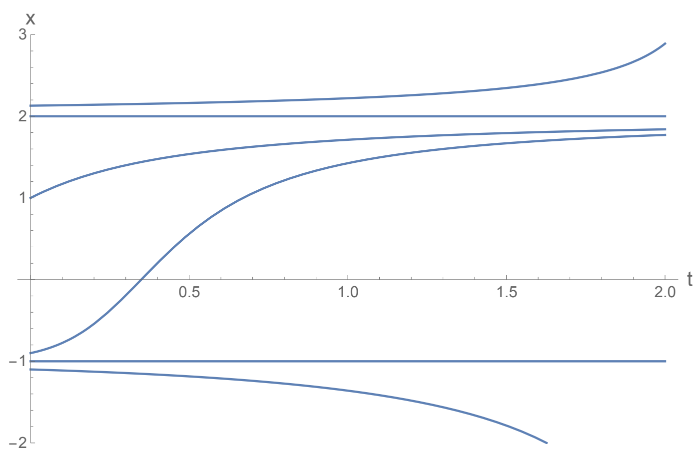
Mathematical Modeling

Due: September 7, 2018

1. Consider the system $\dot{x} = \sin(x)$.
   (a) Find all fixed points of the flow.
   (b) At which points x does the flow have the greatest velocity to the right?
   (c) Find the flows acceleration $\ddot{x}$ as a function of x.
   (d) Find the points where the flow has maximum positive acceleration.

2. For the following equations sketch the vector fields on the real line, if possible find all fixed points, classify their stability and sketch the graph of $x(t)$ for different initial conditions.
   (a) $\dot{x} = 1 - x^{14}$.
   (b) $\dot{x} = e^{-x}\sin(x)$.
   (c) $\dot{x} = 1 - 2\cos(x)$.
   (d) $\dot{x} = e^{x} - \cos(x)$. (You won't be able to find the fixed points explicitly, but you can still determine the qualitative behavior).

3. The curves $x(t)$ illustrated below correspond to solution curves for the differential equation $\dot{x} = f(x)$.



   (a) Sketch a one dimensional phase portrait that is consistent with this figure.
   (b) Sketch a graph of $f(x)$ that is consistent with this figure.
   (c) Give a formula for $f(x)$ that is consistent with this figure.

4. For each of parts (a)-(e), find an equation $\dot{x} = f(x)$ with the stated properties, or if there are no examples, explain why not.
   (a) Every real number is a fixed point.

(b) Every integer is a fixed point, and there are no others.

(c) There are precisely three fixed points, and there are no others.

(d) There are precisely three fixed points, and all of them are stable.

(e) There are no fixed points.

(f) There are precisely 100 fixed points.

5. The velocity $v(t)$ of a skydiver falling to the ground is governed by the equation $m\dot{v} = mg - kv^2$, where m is the mass of the skydiver, g is acceleration due to gravity, and $k > 0$ is a constant related to the amount of air resistance.

   (a) Obtain the analytical solution for $v(t)$, assuming that $v(0) = 0$.

   (b) Find the limit of $v(t)$ as $t \to \infty$. This limiting velocity is called the terminal velocity.

   (c) Give a graphical analysis of this problem, and thereby re-derive a formula for the terminal velocity.

6. Suppose X and Y are two species that reproduce exponentially fast: $\dot{X} = aX$ and $\dot{Y} = bY$, respectively, with initial conditions $X_0, Y_0 > 0$ and growth rates $a, b > 0$. Let $x(t) = X(t)/(X(t) + Y(t))$ denote X's share of the total population.

   (a) Show that $\dot{x} = (a - b)x(1 - x)$.

   (b) Show that if $a > b$ then x is monotonically increasing and approaches 1 as $t \to \infty$. What does this result imply about the population?

   (c) Show that if $a < b$ then x is monotonically decreasing and approaches 0 as $t \to \infty$. What does this result imply about the population?

   (d) What happens if $a = b$?

7. Consider the differential equation

$$\dot{x} = -\frac{dV}{dx},$$

   with $x \in \mathbb{R}$, where the potential $V(x)$ is drawn below. Sketch a phase portrait for this system.