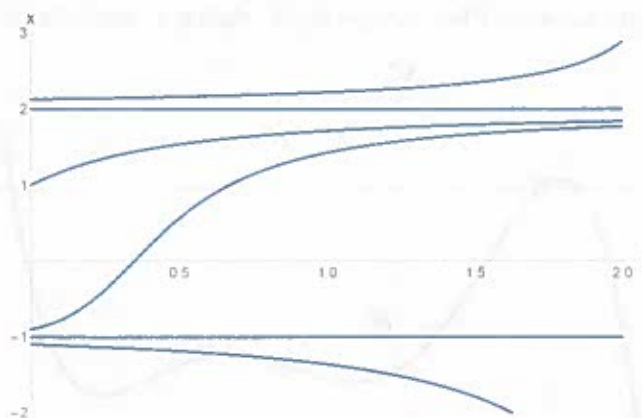


Homework 1

Mathematical Modeling

Due: September 7, 2018

1. Consider the system $\dot{x} = \sin(x)$.
 - (a) Find all fixed points of the flow.
 - (b) At which points x does the flow have the greatest velocity to the right?
 - (c) Find the flow's acceleration \ddot{x} as a function of x .
 - (d) Find the points where the flow has maximum positive acceleration.
2. For the following equations sketch the vector fields on the real line, if possible find all fixed points, classify their stability and sketch the graph of $x(t)$ for different initial conditions.
 - (a) $\dot{x} = 1 - x^{14}$.
 - (b) $\dot{x} = e^{-x} \sin(x)$.
 - (c) $\dot{x} = 1 - 2 \cos(x)$.
 - (d) $\dot{x} = e^x - \cos(x)$. (You won't be able to find the fixed points explicitly, but you can still determine the qualitative behavior).
3. The curves $x(t)$ illustrated below correspond to solution curves for the differential equation $\dot{x} = f(x)$.

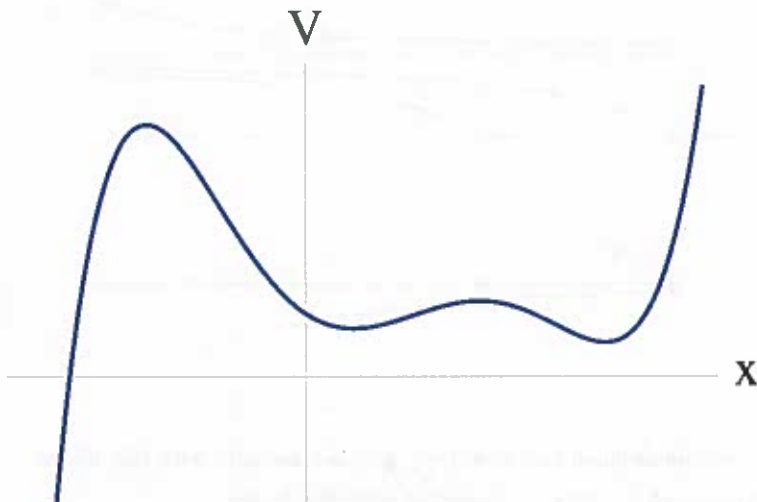


- (a) Sketch a one dimensional phase portrait that is consistent with this figure.
 - (b) Sketch a graph of $f(x)$ that is consistent with this figure.
 - (c) Give a formula for $f(x)$ that is consistent with this figure.
4. For each of parts (a)-(c), find an equation $\dot{x} = f(x)$ with the stated properties, or if there are no examples, explain why not.
 - (a) Every real number is a fixed point.

- (b) Every integer is a fixed point, and there are no others.
 - (c) There are precisely three fixed points, and there are no others.
 - ✓(d) There are precisely three fixed points, and all of them are stable.
 - ✓(e) There are no fixed points.
 - ✓(f) There are precisely 100 fixed points.
5. The velocity $v(t)$ of a skydiver falling to the ground is governed by the equation $m\dot{v} = mg - kv^2$, where m is the mass of the skydiver, g is acceleration due to gravity, and $k > 0$ is a constant related to the amount of air resistance.
- (a) Obtain the analytical solution for $v(t)$, assuming that $v(0) = 0$.
 - (b) Find the limit of $v(t)$ as $t \rightarrow \infty$. This limiting velocity is called the terminal velocity.
 - (c) Give a graphical analysis of this problem, and thereby re-derive a formula for the terminal velocity.
6. Suppose X and Y are two species that reproduce exponentially fast: $\dot{X} = aX$ and $\dot{Y} = bY$, respectively, with initial conditions $X_0, Y_0 > 0$ and growth rates $a, b > 0$. Let $x(t) = X(t)/(X(t) + Y(t))$ denote X 's share of the total population.
- (a) Show that $\dot{x} = (a - b)x(1 - x)$.
 - (b) Show that if $a > b$ then x is monotonically increasing and approaches 1 as $t \rightarrow \infty$. What does this result imply about the population?
 - (c) Show that if $a < b$ then x is monotonically decreasing and approaches 0 as $t \rightarrow \infty$. What does this result imply about the population?
 - (d) What happens if $a = b$?
7. Consider the differential equation

$$\dot{x} = -\frac{dV}{dx},$$

with $x \in \mathbb{R}$, where the potential $V(x)$ is drawn below. Sketch a phase portrait for this system.



Solutions:

#2. Graphically analyze the following systems

a) $\dot{x} = 1 - x^2$

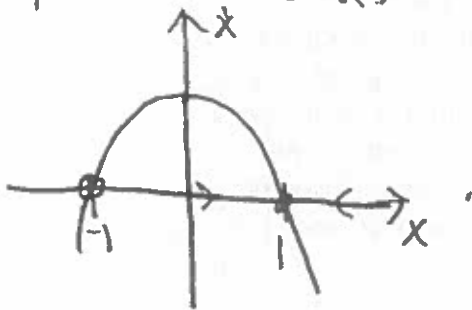
b) $\dot{x} = e^{-x} \sin(x)$

c) $\dot{x} = 1 - 2\cos(x)$

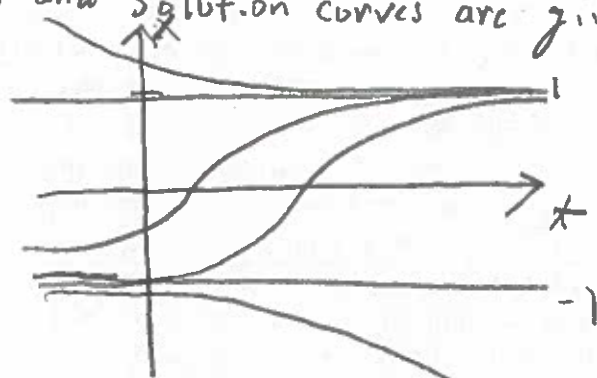
d) $\dot{x} = e^x - \cos(x)$

Solutions:

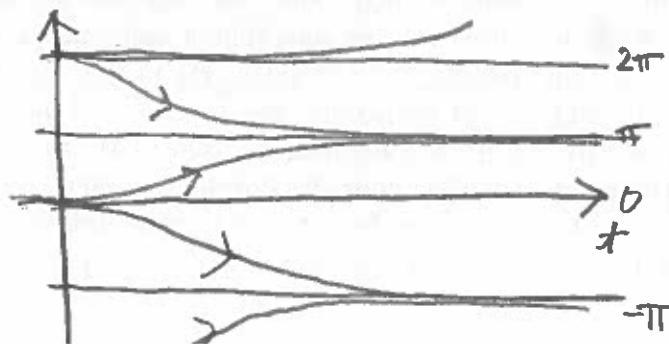
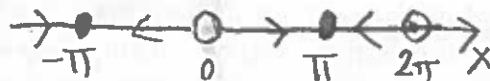
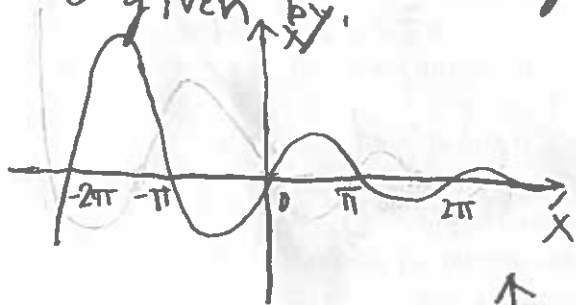
a.) Fixed points are given by $x^* = \pm 1$. To determine stability we plot \dot{x} versus x .



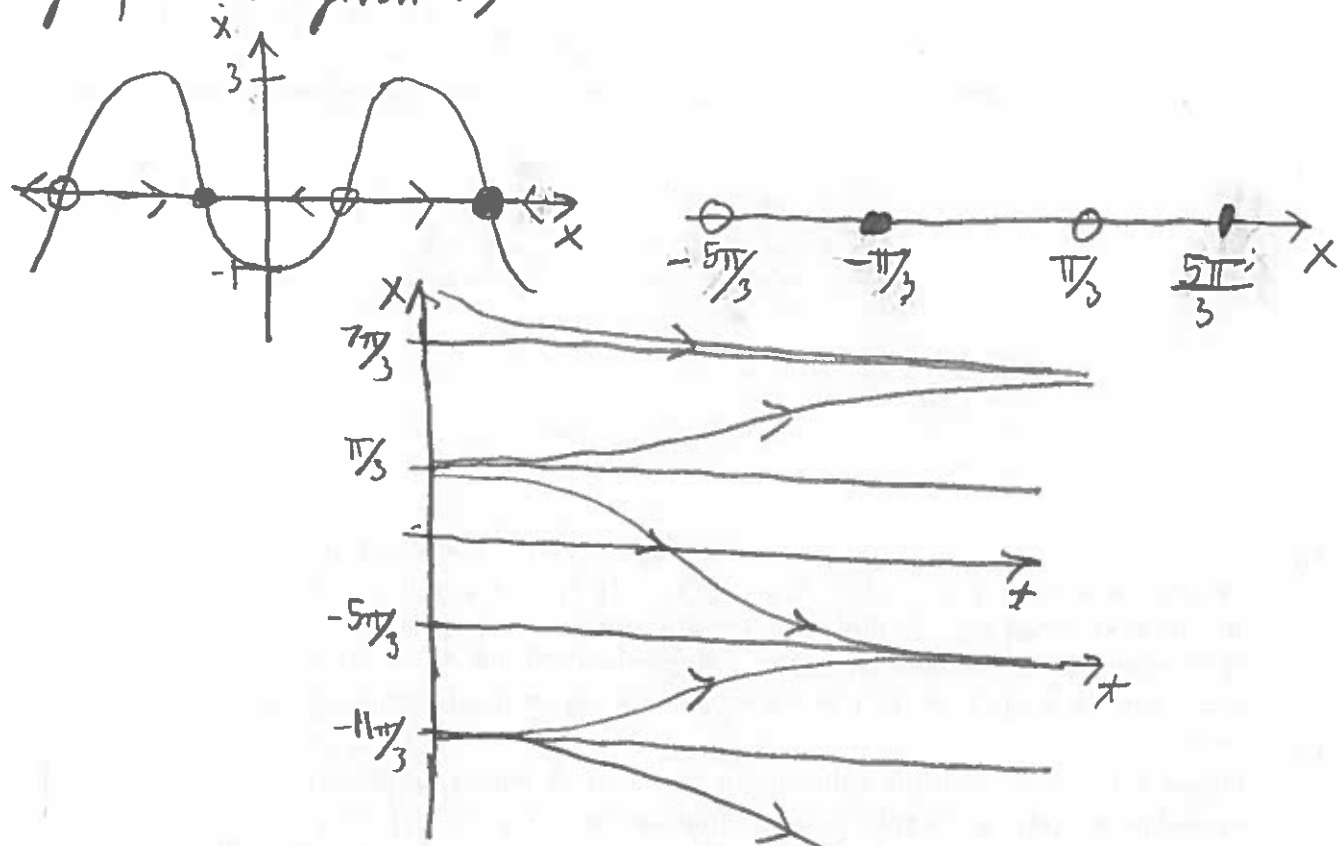
Consequently, the phase portrait and solution curves are given by:



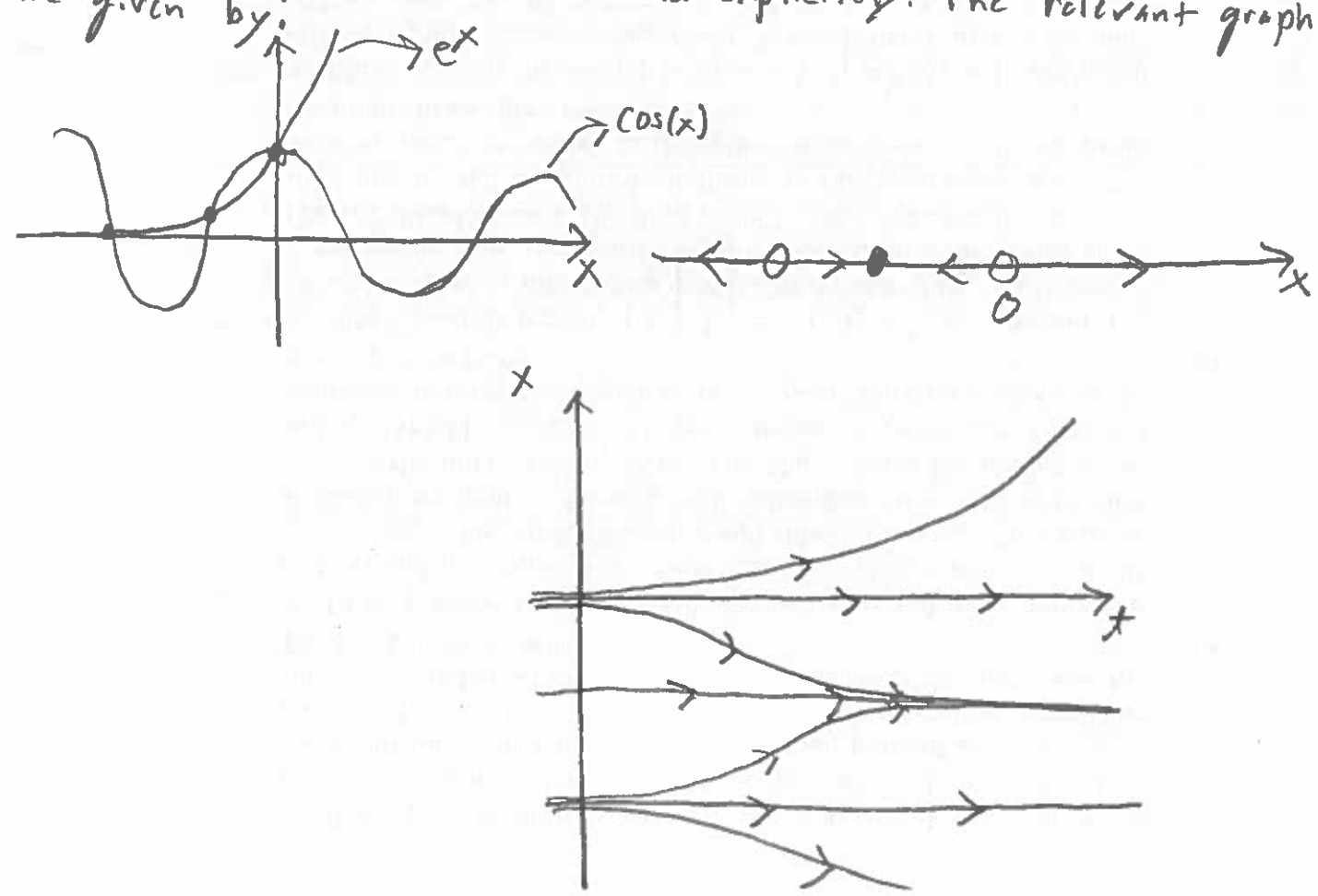
b.) The fixed points are given by $x^* = n\pi$. The relevant graphs are given by:



c) The fixed points are given by $x^* = \pm \frac{\pi}{3} + 2n\pi$. The relevant graphs are given by:



d) The fixed points cannot be found explicitly. The relevant graphs are given by:



#3.

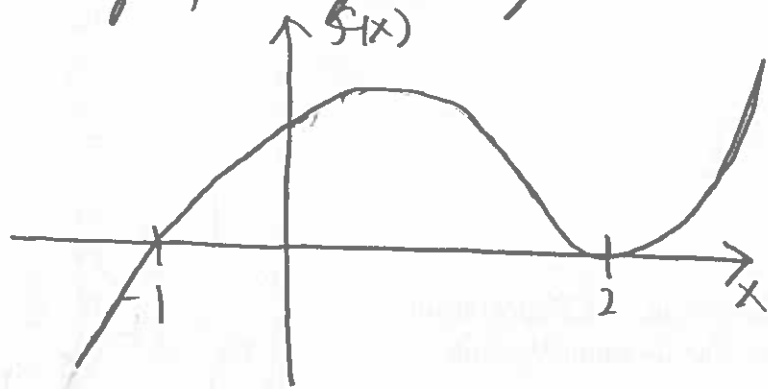
a.) Sketch a one dimensional phase portrait that is consistent with this figure.

Solution:

The phase portrait is given by:



b.) Sketch a graph of $f(x)$ that is consistent with this figure.
The graph is given by:



c.) A possible formula for $f(x)$ is given by:

$$f(x) = -(x+1)(x-2)^2.$$

#4.

Find an equation $\dot{x} = f(x)$ with the stated properties.

a.) Every real number is a fixed point.

Solution:

$$\dot{x} = 0.$$

d.) There are precisely three fixed points and all of them are stable.

Solution:

This is not possible for a smooth f since between two stable fixed points there must exist an unstable fixed point.

e.) There are no fixed points.

Solution:

$$\dot{x} = 1.$$

f.) There are precisely 100 fixed points.

Solution:

$$\dot{x} = \prod_{i=1}^{100} (x - i).$$

#5.

The velocity $v(t)$ of a skydiver is given by $m\dot{v} = mg - kv^2$, where m is the mass of the skydiver, g is the acceleration due to gravity, and $k > 0$ is a constant related to the amount of air resistance.

a.) Obtain an analytical solution for $v(t)$, assuming $v(0) = 0$.

Solution:

Separating variables it follows that:

$$m \frac{dv}{dt} = mg - kv^2$$

$$\Rightarrow \int_0^v \frac{1}{g - \frac{k}{m}v^2} dv = t$$

$$\Rightarrow \frac{1}{g} \int_0^v \frac{1}{(1 - \sqrt{\frac{k}{gm}}v)(1 + \sqrt{\frac{k}{gm}}v)} dv = t$$

$$\Rightarrow \frac{1}{g} \int_0^v \frac{1}{2(1 - \sqrt{\frac{k}{gm}}v)} dv + \int_0^v \frac{1}{2(1 + \sqrt{\frac{k}{gm}}v)} dv = t$$

$$\Rightarrow \frac{1}{2} \sqrt{\frac{m}{kg}} \ln \left(\frac{1 + \sqrt{\frac{k}{gm}}v}{1 - \sqrt{\frac{k}{gm}}v} \right) = t$$

$$\Rightarrow \frac{1 + \sqrt{\frac{k}{gm}}v}{1 - \sqrt{\frac{k}{gm}}v} = e^{2\sqrt{\frac{kg}{m}}t}$$

$$\Rightarrow v(t) = \sqrt{\frac{mg}{k}} \frac{\exp(2\sqrt{\frac{kg}{m}}t) - 1}{\exp(2\sqrt{\frac{kg}{m}}t) + 1}$$

b.) Find the limit of $v(t)$ as $t \rightarrow \infty$. This limiting velocity is called the terminal velocity.

Solution:

Calculating it follows that:

$$\lim_{t \rightarrow \infty} v(t) = \sqrt{\frac{mg}{k}}$$

c.) Give a graphical analysis of the problem.

Solution:

The phase portrait for this system is given by:



Consequently, $\lim_{t \rightarrow \infty} v(t) = \sqrt{\frac{mg}{k}}$.

#6.

Suppose X and Y are two species that reproduce exponentially fast $\dot{X} = aX$, $\dot{Y} = bY$. Let x denote X 's share of the total population.

a.) Show that $\dot{x} = (a-b)x(1-x)$.

b.) Show that if $a > b$ then x is monotonically increasing and approaches 1 as $t \rightarrow \infty$.

c.) Show that if $a < b$ then x is monotonically decreasing and approaches 0 as $t \rightarrow \infty$.

d.) What happens if $a = b$?

Solution:

a.) Differentiating it follows that

$$\dot{x} = \frac{d}{dt} \frac{X}{X+Y} = \frac{(X+Y)\dot{X} - X(\dot{X} + \dot{Y})}{(X+Y)^2}$$

$$\Rightarrow \dot{x} = \frac{(X+Y)aX - X(aX + bY)}{(X+Y)^2}$$

$$\Rightarrow \dot{x} = \frac{(a-b)XY}{(X+Y)^2}$$

$$\Rightarrow \dot{x} = (a-b)x(1-x)$$

b-c) Since $x(0) = x_0 < 1$, if $a > b$ ($a < b$) then $\dot{x} > 0$ ($\dot{x} < 0$) which implies x is monotonically increasing (decreasing).

d.) If $a = b$ then $\dot{x} = 0$ which implies that the ratio of populations remains constant.