Homework 2

Mathematical Modeling

Due: September 14, 2018

1 Problems for Everybody

1. D'Arcy Wentworth Thompson, a noted scientist of natural history, wrote in his book, *On Growth and Form* (1917): "But why, in the general run of shells, all the world over, in the past and in the present, one direction of twist is so overwhelmingly commoner than the other, no man knows." Most snails species are *dextral* (right handed) in their shell pattern. *Sinistral* (left-handed) snails are exceedingly rare.

   (a) Let $p(t)$ be the ratio of dextral snails in the population of snails. Explain why
   $$\frac{dp}{dt} = rp(1-p)\left(p - \frac{1}{2}\right),$$
   $$p(0) = p_0$$
   is a plausible model for the dynamics of dextral snails if we assume $0 < p_0 < 1$.

   (b) Sketch a phase portrait for this system.

   (c) Suppose $p_0 \approx 1/2$. Explain in practical terms why this phase portrait justifies the observation that sinistral snails are rare. Explain why this is essentially a fluke and we could just as easily be debating why dextral snails are rare.

2. In class we developed the logistic growth model of population growth:
   $$\frac{dP}{dt} = rP\left(1 - \frac{P}{\kappa}\right),$$
   $$P(0) = P_0,$$
   where $r > 0$ is a growth rate and $\kappa > 0$ is a carrying capacity. This model has unstable and stable fixed points at $P = 0$ and $P = \kappa$ respectively. Therefore, the model predicts that for for all positive initial conditions the population will reach equilibrium at the carrying capacity κ. However, this is somewhat unrealistic. Suppose we wanted to model population growth of humans and we start with $P_0 = 1$. Clearly the population would die off. In particular, for a sufficiently small initial population we would expect the population to die off. In this problem we are going to a develop a new mathematical model of the form
   $$\frac{dP}{dt} = F(P)$$
   that corrects this problem in the logistic model.

   (a) What properties should $F(P)$ satisfy in order to represent a realistic model of population growth? Your function must account for population extinction if P_0 is sufficiently small. Justify your answer.

   (b) Sketch a graph of $F(P)$. Be sure to label everything that is important for the model.

   (c) Give a possible analytic formula for F that satisfies the properties you outlined above. With modeling you want to give the simplest possible example that works.

(d) Sketch a phase portrait for you system and discuss the consequences of this model.

3. For each of the following problems sketch all qualitatively different phase portraits that occur as r is varied. Sketch a bifurcation diagram of fixed points x^* versus r. In each bifurcation diagram determine what type of bifurcation occurs.

   (a) $\dot{x} = 1 + rx + x^2$.

   (b) $\dot{x} = rx + x^2$.

   (c) $\dot{x} = r - \cosh(x)$.

   (d) $\dot{x} = x - rx(1 - x)$.

   (e) $\dot{x} = x + \frac{rx}{1+x^2}$.

   (f) $\dot{x} = r - 3x^2$.

   (g) $\dot{x} = rx - \frac{x}{1+x^2}$.

   (h) $\dot{x} = rx + \frac{x^3}{1+x^2}$.

4. In class we developed two different models with harvesting, i.e. fish in a lake. We will now develop and analyze a third. Consider the following mathematical model of fish in a lake:

$$\frac{dP}{dt} = F(P) - \rho P,$$
$$P(0) = P_0,$$

   where F is the growth model you constructed in problem #2 and $\rho > 0$ is a constant.

   (a) What does the term $-\rho P$ represent in practical terms?

   (b) Sketch a bifurcation diagram for this problem. What does this diagram tell you in practical terms?

2 Problems for MST 651 students only. Students in MST 351 can complete these problems for extra credit

1. Kermack and McKendrick (1927) proposed the following simple model for the evolution of an epidemic. Suppose the population can be divided into three classes: $x(t)$ = number of healthy people; $y(t)$ = number of sick people; $z(t)$ = number of dead people. Assume that total population remains constant in size, except for deaths due to the epidemic. Then the model is

$$\dot{x} = -kxy$$
$$\dot{y} = kxy - ly$$
$$\dot{z} = ly$$

   where $l, k > 0$ are constants.

   (a) Explain in practical terms what each term in this equation represents.

   (b) Show that $x + y + z = N$, where N is a constant.

   (c) Use the $\dot{x}$ and $\dot{z}$ equation to show that $x(t) = x_0 \exp(-kz(t)/l)$, where $x_0 = x(0)$. Hint: Consider the ratio $\dot{x}/\dot{z}$.

   (d) Show that $\dot{z}$ satisfies the first order equation $\dot{z} = l(N - z - z_0 \exp(-kz/l))$.

   (e) Show that this equation can be nondimensionalized to

$$\frac{du}{d\tau} = a - bu - e^{-u}.$$

   (f) Show that $a \geq 1$ and $b > 0$.

(g) Determine the number of fixed points u^* and classify their stability.

(h) Show that maximum of $\dot{u}$ occurs at the same time as the maximum of both $\dot{z}(t)$ and $y(t)$. This time is called the peak of the epidemic and is denoted t_{peak}.

(i) Show that if $b < 1$, then $\dot{u}$ is increasing at $t = 0$ and reaches its maximum at time t_{peak}. Show that $\dot{u}$ eventually decreases to zero.

(j) Show that if $b > 1$ then $t_{\text{peak}} = 0$, i.e. no epidemic occurs.

(k) Give a biological interpretation of the constant b.