Homework 2

Mathematical Modeling

Due: September 14, 2018

1 Problems for Everybody

1. D'Arcy Wentworth Thompson, a noted scientist of natural history, wrote in his book, *On Growth and Form* (1917): "But why, in the general run of shells, all the world over, in the past and in the present, one direction of twist is so overwhelmingly commoner than the other, no man knows." Most snails species are *dextral* (right handed) in their shell pattern. *Sinistral* (left-handed) snails are exceedingly rare.

   (a) Let $p(t)$ be the ratio of dextral snails in the population of snails. Explain why

   $$\frac{dp}{dt} = rp(1-p)\left(p - \frac{1}{2}\right),$$
   $$p(0) = p_0$$

   is a plausible model for the dynamics of dextral snails if we assume $0 < p_0 < 1$.

   (b) Sketch a phase portrait for this system.

   (c) Suppose $p_0 \approx 1/2$. Explain in practical terms why this phase portrait justifies the observation that sinistral snails are rare. Explain why this is essentially a fluke and we could just as easily be debating why dextral snails are rare.

2. In class we developed the logistic growth model of population growth:

   $$\frac{dP}{dt} = rP\left(1 - \frac{P}{\kappa}\right),$$
   $$P(0) = P_0,$$

   where $r > 0$ is a growth rate and $\kappa > 0$ is a carrying capacity. This model has unstable and stable fixed points at $P = 0$ and $P = \kappa$ respectively. Therefore, the model predicts that for for all positive initial conditions the population will reach equilibrium at the carrying capacity κ. However, this is somewhat unrealistic. Suppose we wanted to model population growth of humans and we start with $P_0 = 1$. Clearly the population would die off. In particular, for a sufficiently small initial population we would expect the population to die off. In this problem we are going to a develop a new mathematical model of the form

   $$\frac{dP}{dt} = F(P)$$

   that corrects this problem in the logistic model.

   (a) What properties should $F(P)$ satisfy in order to represent a realistic model of population growth? Your function must account for population extinction if P_0 is sufficiently small. Justify your answer.

   (b) Sketch a graph of $F(P)$. Be sure to label everything that is important for the model.

   (c) Give a possible analytic formula for F that satisfies the properties you outlined above. With modeling you want to give the simplest possible example that works.

(d) Sketch a phase portrait for you system and discuss the consequences of this model.

3. For each of the following problems sketch all qualitatively different phase portraits that occur as r is varied. Sketch a bifurcation diagram of fixed points x^* versus r. In each bifurcation diagram determine what type of bifurcation occurs.

   (a) $\dot{x} = 1 + rx + x^2$.

   (b) $\dot{x} = rx + x^2$.

   (c) $\dot{x} = r - \cosh(x)$.

   (d) $\dot{x} = x - rx(1 - x)$.

   (e) $\dot{x} = x + \frac{rx}{1+x^2}$.

   (f) $\dot{x} = r - 3x^2$.

   (g) $\dot{x} = rx - \frac{x}{1+x^2}$.

   (h) $\dot{x} = rx + \frac{x^3}{1+x^2}$.

4. In class we developed two different models with harvesting, i.e. fish in a lake. We will now develop and analyze a third. Consider the following mathematical model of fish in a lake:

$$\frac{dP}{dt} = F(P) - \rho P,$$
$$P(0) = P_0,$$

where F is the growth model you constructed in problem #2 and $\rho > 0$ is a constant.

   (a) What does the term $-\rho P$ represent in practical terms?

   (b) Sketch a bifurcation diagram for this problem. What does this diagram tell you in practical terms?

2 Problems for MST 651 students only. Students in MST 351 can complete these problems for extra credit

1. Kermack and McKendrick (1927) proposed the following simple model for the evolution of an epidemic. Suppose the population can be divided into three classes: $x(t)$ = number of healthy people; $y(t)$ = number of sick people; $z(t)$ = number of dead people. Assume that total population remains constant in size, except for deaths due to the epidemic. Then the model is

$$\dot{x} = -kxy$$
$$\dot{y} = kxy - ly$$
$$\dot{z} = ly$$

where $l, k > 0$ are constants.

   (a) Explain in practical terms what each term in this equation represents.

   (b) Show that $x + y + z = N$, where N is a constant.

   (c) Use the $\dot{x}$ and $\dot{z}$ equation to show that $x(t) = x_0 \exp(-kz(t)/l)$, where $x_0 = x(0)$. Hint: Consider the ratio $\dot{x}/\dot{z}$.

   (d) Show that $\dot{z}$ satisfies the first order equation $\dot{z} = l(N - z - z_0 \exp(-kz/l))$.

   (e) Show that this equation can be nondimensionalized to

$$\frac{du}{d\tau} = a - bu - e^{-u}.$$

   (f) Show that $a \geq 1$ and $b > 0$.

(g) Determine the number of fixed points u^* and classify their stability.

(h) Show that maximum of $\dot{u}$ occurs at the same time as the maximum of both $\dot{z}(t)$ and $y(t)$. This time is called the peak of the epidemic and is denoted $t_{\mathbf{peak}}$.

(i) Show that if $b < 1$, then $\dot{u}$ is increasing at $t = 0$ and reaches its maximum at time $t_{\mathbf{peak}}$. Show that $\dot{u}$ eventually decreases to zero.

(j) Show that if $b > 1$ then $t_{\mathbf{peak}} = 0$, i.e. no epidemic occurs.

(k) Give a biological interpretation of the constant b.

Problems For Everybody

#1.

a) Let $p(t)$ be the ratio of dextral snails in the popolation of snails. Explain why

$$\frac{dp}{dt} = rp(1-p)(p-\tfrac{1}{2}) \quad *$$

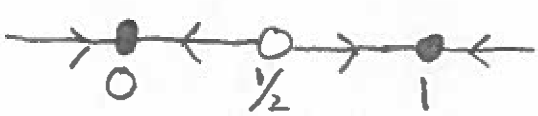
is a plausible model for the dynamics of dextral snails if we assume $0<p_0<1$.

Solution:

Let $f(p)=rp(1-p)(p-\tfrac{1}{2})$. The roots of f are given by $p=0, \tfrac{1}{2}, 1$. The roots $p=0,1$ make sense in that if the population contains no dextral snails ($p=0$) or all dextral snails ($p=1$) the ratio of dextral to senstral snails will not change in time. The root $p=\tfrac{1}{2}$ models that if the ratio to dextral to senstral snails is precisely unity the proportion of dextral to senstral snails will not change in time. Moreover, f is chosen so that $p=0,1$ are stable and $p=\tfrac{1}{2}$ is unstable. This is plausible since if there is any imbalance of population, i.e $p>\tfrac{1}{2}$ or $p<\tfrac{1}{2}$, one of the population types will be selected.

b) Sketch a phase portrait for this system.

Solution:



c) Suppose $p_0 \approx \tfrac{1}{2}$. Explain in practical terms why this phase portrait justifies the observation that sinstral snails are rare.

Solution:

Suppose $p_0=\tfrac{1}{2}+\varepsilon$, where $|\varepsilon|<<1$ and ε can be negative. Then solution to $*$ satisfy

$$\lim_{t\to\infty} p(t) = \begin{cases} 1 & \text{if } \varepsilon>0 \\ 0 & \text{if } \varepsilon<0 \end{cases}.$$

Consequently, the sign of ε, i.e. the measure of population imbalance, dictates the longterm equilibrium population. That is, the slight imbalance of population determines whether dextral or sinistral snails dominate.

#2.

a) What properties should $F(p)$ satisfy in order to represent a realistic model of population growth?

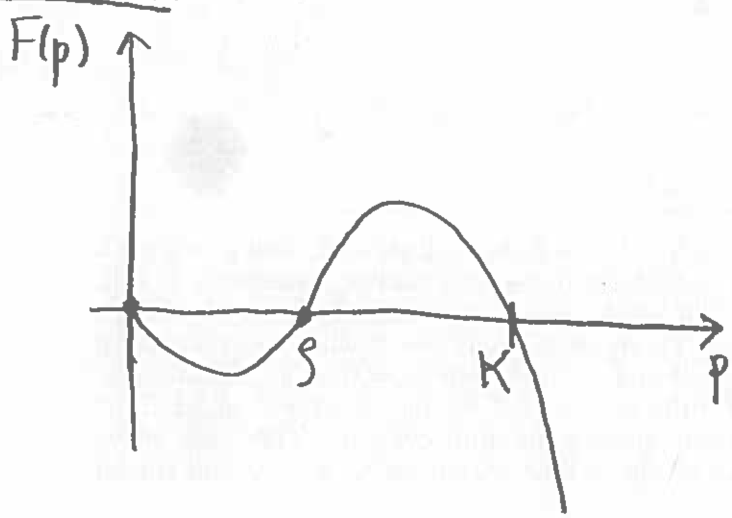
Solution:

The simplest properties are as follows:

1. $F(0)=0$.
2. There exists $K \geq \rho > 0$ such that $F(K)=F(\rho)=0$.
3. $\lim_{p \to \infty} F(p) = -\infty$.

b) Sketch a graph of $F(p)$. Be sure to label everything that is important for the model.

Solution:



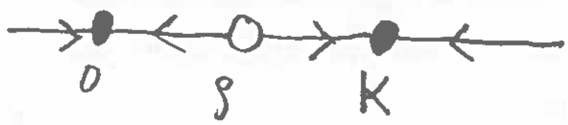
c) Give a possible analytic formula for F that satisfies the properties outlined above.

Solution:

$$F(p) = -rp\left(1 - \frac{p}{\rho}\right)\left(1 - \frac{p}{K}\right).$$

d) Sketch a phase portrait for your system and discuss the consequences of this model.

Solution:



This model predicts the population will either go extinct or converge to the carraying capacity K depending on whether $p_0 > \rho$.

#3.

For each of the following problems sketch all qualitatively different phase portraits that occur as r is varied.

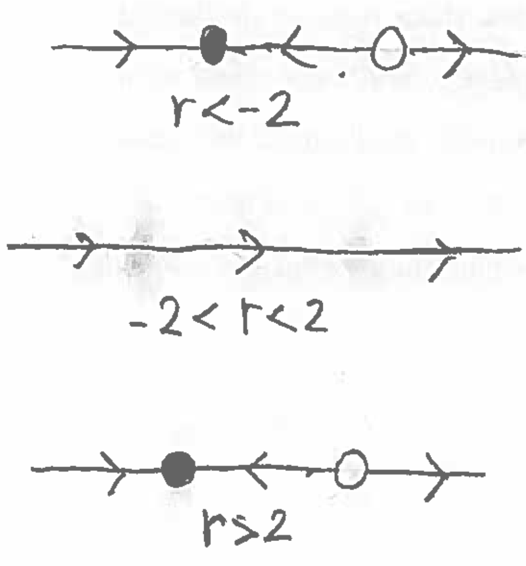
a) $\dot{x} = 1 + rx + x^2$

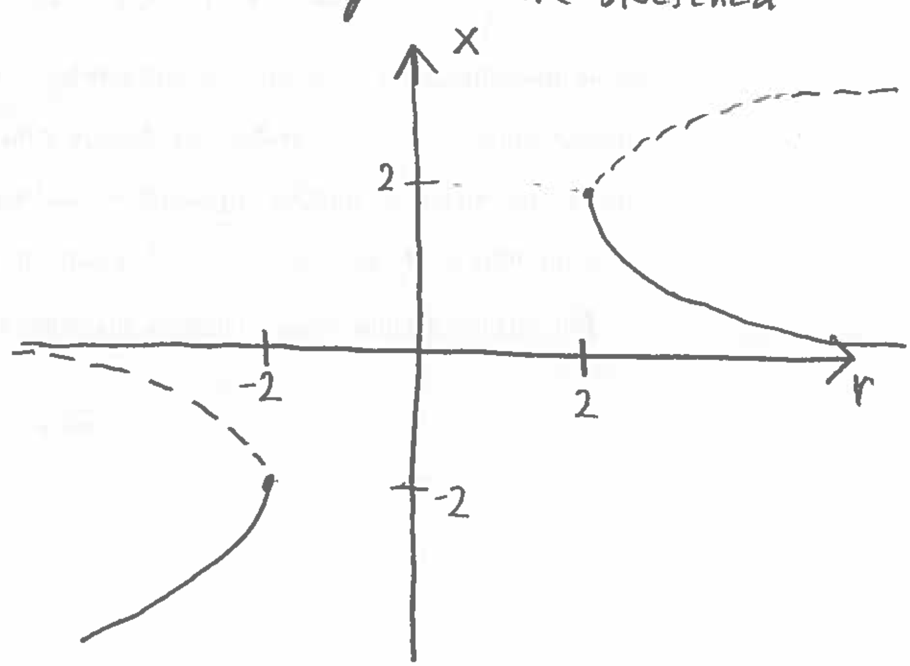
Solution:

The fixed points are given by:

$$x = \frac{-r \pm \sqrt{r^2 - 4}}{2}.$$

Consequently, there are saddle node bifurcations at $r = \pm 2$. The phase portraits and bifurcation diagrams are sketched below:





e) $\dot{x} = x + \dfrac{rx}{1+x^2}$.

<u>Solution:</u>

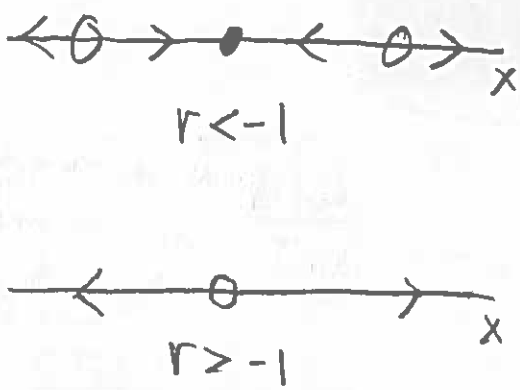
The fixed points satisfy:

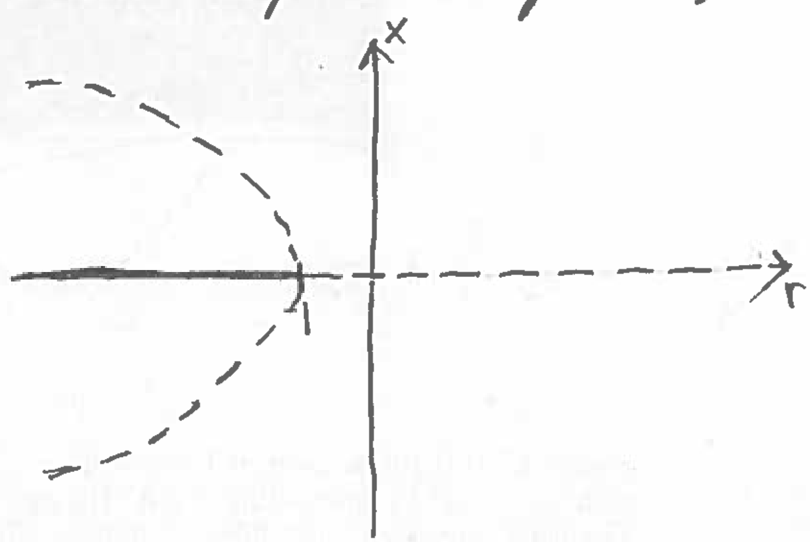
$$x + \frac{rx}{1+x^2} = 0$$

$$\Rightarrow x = 0,\ r + 1 + x^2 = 0$$

$$\Rightarrow x = 0,\ x = \pm\sqrt{-r-1}.$$

At $r=-1$ there is a bifurcation. Since $\lim\limits_{x\to\infty} x + \dfrac{rx}{1+x^2} = \infty$ it follows that the rightmost fixed point is unstable. Consequently, the phase portraits and bifurcation diagram are given by:





g) $\dot{x} = rx - \dfrac{x}{1+x^2}$

<u>Solution:</u>

The fixed points satisfy:

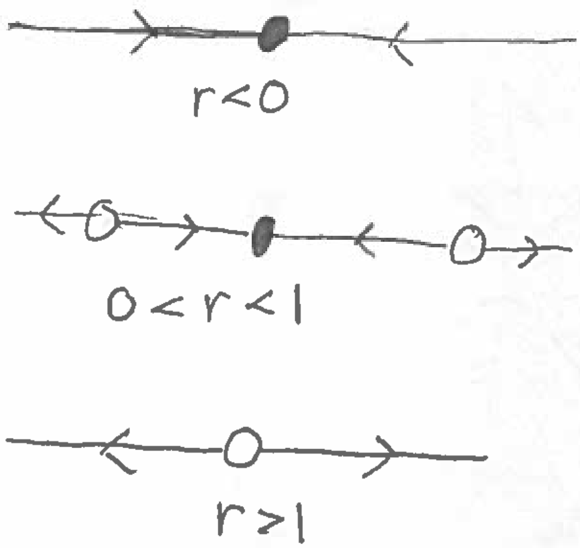
$$x = 0,\ r + rx^2 - 1 = 0$$

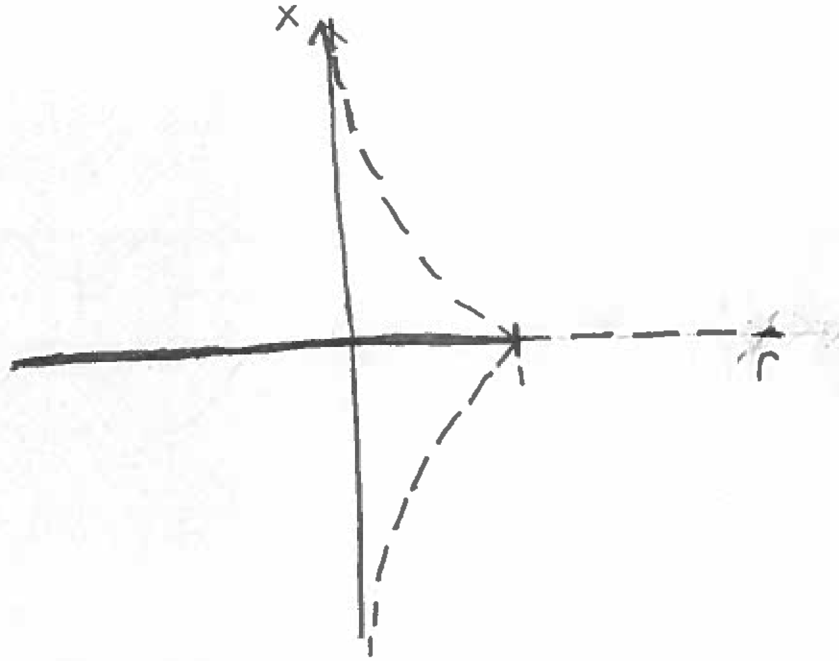
$$\Rightarrow x = 0,\ x = \pm\sqrt{\frac{1}{r} - 1}$$

At $r=1$ there is a bifurcation and three fixed points exist if $0<r<1$. There is another bifurcation at $r=0$. Moreover,

$$\lim_{x\to\infty} rx - \frac{x}{1+x^2} = \begin{cases} \infty, & \text{if } r>0 \\ -\infty, & \text{if } r<0 \end{cases}$$

which implies the rightmost fixed point is unstable if $r>0$ and is stable if $r<0$. Therefore, the phase portraits and bifurcation diagrams are given by:





h) $\dot{x} = rx + \frac{x^3}{1+x^2}$.

Solution:

The fixed points satisfy:

$$x = 0, \quad r(1+x^2) + x^2 = 0$$

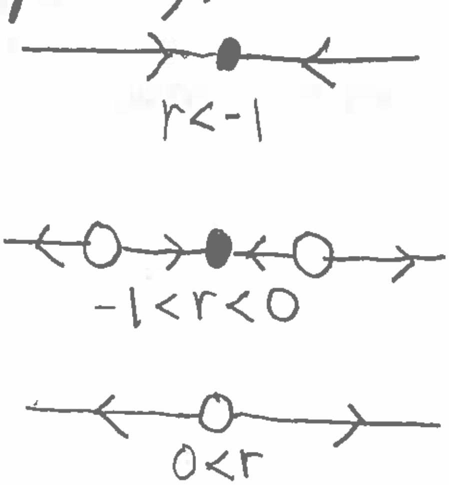
$$\Rightarrow x = 0, \quad x^2 = \frac{-r}{1+r}$$

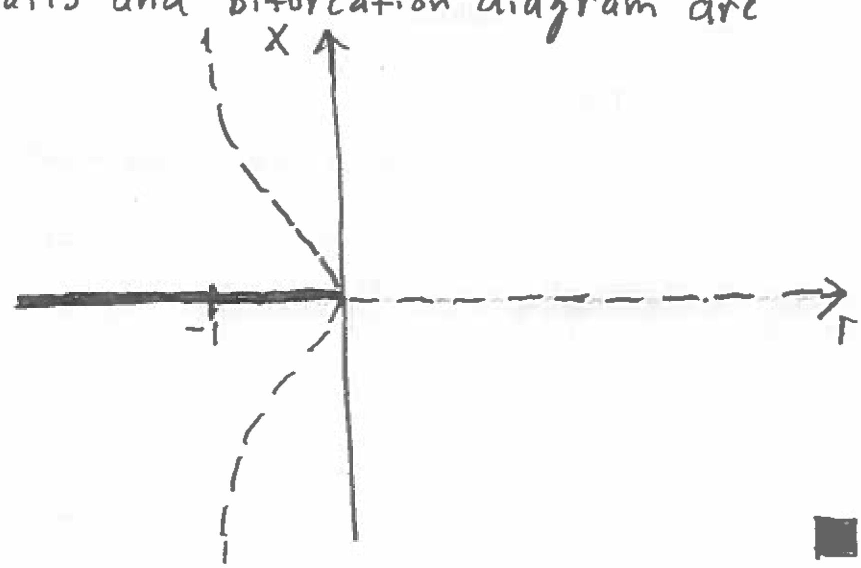
$$\Rightarrow x = 0, \quad x = \pm\sqrt{\frac{-r}{1+r}}$$

Consequently, there are bifurcations at $r = 0, -1$. Moreover,

$$\lim_{x\to\infty} rx + \frac{x^3}{1+x^2} = \begin{cases} \infty & \text{if } r > -1 \\ -\infty & \text{if } r < -1 \end{cases}$$

Therefore, the phase portraits and bifurcation diagram are given by:





#4.

Consider the mathematical model of fish in a lake:

$$\frac{dP}{dt} = F(P) - \rho P,$$
$$P(0) = P_0$$

where F is the growth model you constructed in problem #2 and $\rho > 0$ is a constant.

a.) What does the term $-\rho P$ represent in practical terms?

Solution:

This term represents harvesting of fish proportional to population size.

b.) Sketch a bifurcation diagram for this problem. What does this diagram tell you in practical terms?

Solution:

The system is given by:

$$\frac{dP}{dt} = -rP(1 - P/\alpha)(1 - P/K) - \rho P.$$

Rescale variables by $\tau = rt$, $x = P/K$. Therefore,

$$rK\frac{dx}{d\tau} = -rKx(1 - \gamma x)(1 - x) - \rho K x$$

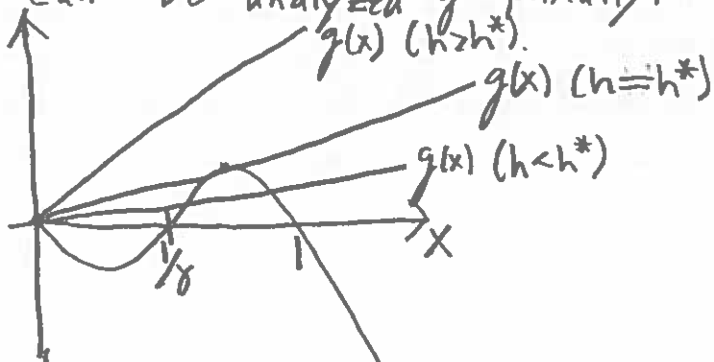
$$\Rightarrow \frac{dx}{d\tau} = -x(1 - \gamma x)(1 - x) - hx$$

where $\gamma = \alpha/K$ is the ratio of the carrying capacity to the intermediate population size and $h = \rho/r$ is the ratio of the harvesting rate to the growth rate.

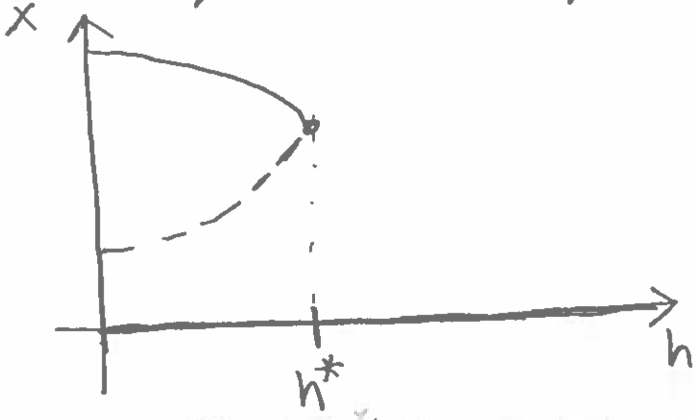
To determine the bifurcation we investigate the intersection of the functions:

$$f(x) = -x(1 - \gamma x)(1 - x) \text{ and } g(x) = hx.$$

This can be analyzed graphically!



The resulting bifurcation diagram is then:



Consequently, if $h>h^*$ the population goes extinct. ■

MST 651 Problem:

#1

a.) Explain in practical terms what each term in this equation represents.

Solution:

K is an infection rate per population while ℓ is a death rate.

b.) Show that $x+y+z=N$, where N is a constant.

Solution:

Calculating it follows that

$$\dot{N}=\dot{x}+\dot{y}+\dot{z}=-kxy+kxy-\ell y+\ell y=0.$$

Therefore, N is constant.

c.) Use the $\dot{x}$ and $\dot{z}$ equation to show that $x(t)=x_0\exp(-kz(t)/\ell)$.

Solution:

$$\frac{\dot{x}}{\dot{z}}=\frac{dx}{dz}=-\frac{kx}{\ell}$$

Therefore,

$$x(z)=x_0\exp(-kz(t)/\ell).$$

d.) Show that $\dot{z}$ satisfies $\dot{z}=\ell(N-z-x_0\exp(-kz/\ell))$.

Solution:

Since $x+y+z=0$ it follows that

$$\dot{z}=\ell y=\ell(N-z-x)$$

$$\Rightarrow \dot{z}=\ell(N-z-x_0\exp(-kz/\ell)).$$

e.) Show that this equation be nondimensionalized to

$$\frac{du}{d\tau} = a - bu - e^{-u}$$

Solution:

Let $u = Kz/\ell$. Then

$$\frac{\ell}{K}\dot{u} = \ell\left(N - \frac{\ell}{K}u - z_0\exp(-u)\right)$$

$$\Rightarrow \dot{u} = KN - \ell u - z_0 K\exp(-u)$$

If we let $\tau = z_0 K t$ then

$$z_0 K\frac{du}{d\tau} = KN - \ell u - z_0 K\exp(-u)$$

$$\Rightarrow \frac{du}{d\tau} = \frac{N}{z_0} - \frac{\ell}{z_0 K}u - e^{-u}$$

$$\Rightarrow \frac{du}{d\tau} = a - bu - e^{-u}.$$

f.) Show that $a \geq 1$ and $b > 0$.

Solution:

Since $a = \frac{N}{z_0} = 1 + \frac{x_0}{z_0} + \frac{y_0}{z_0}$ it follows $a \geq 1$ and $b > 0$ by construction.

g.) Determine the number of fixed points and classify their stability.

Solution:

Let $f(u) = a - bu - e^{-u}$. Therefore,

$$f'(u) = -b + e^{-u} \text{ and } f''(u) = -e^{-u}.$$

Consequently, f is a concave down function satisfying:

$$\lim_{u\to\infty} f(u) = -\infty$$

$$f(0) = a - 1 > 0.$$

Therefore, $\frac{du}{d\tau}$ contains a single fixed point which is stable.

h) Show that the maximum of $\dot{u}$ occurs at the same time as the maximum of $\dot{z}$ and y.

Solution:

Since $u = kz/\ell$ and $\dot{z} = \ell y$ it follows that the maximum of $\dot{u}$ occurs at the maximum of $\dot{z}$ and y.

i) Show that if $b<1$, then $\dot{u}$ is increasing at $t=0$ and reaches a maximum at a time t_{peak}. Show that $\dot{u}$ eventually decreases to zero.

Solution:

Since $u(0)=0$ it follows that:

$$\frac{d^2u}{d\tau^2} = \frac{d}{d\tau}\frac{du}{d\tau} = \frac{d}{d\tau}(a - bu - e^{-u}) = -b\frac{du}{d\tau} + e^{-u}\frac{du}{d\tau}$$

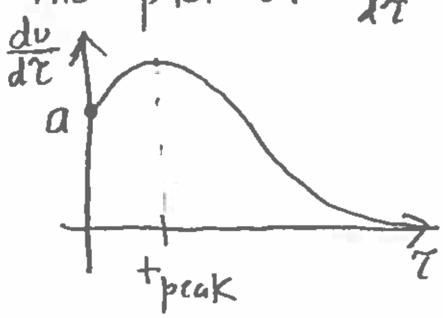
$$\Rightarrow \frac{d^2u}{d\tau^2} = (a - bu - e^{-u})(-b + e^{-u})$$

$$\Rightarrow \left.\frac{d^2u}{d\tau^2}\right|_{\tau=0} = (a-1)(1-b).$$

Consequently, since $a \geq 1$ it follows that if $b<1$ then $\dot{u}$ is increasing. Now since there is a single fixed point it follows that:

1. $\left.\frac{du}{d\tau}\right|_{\tau=0} = a > 0$
2. $\lim\limits_{\tau\to\infty} \frac{du}{d\tau} = 0$

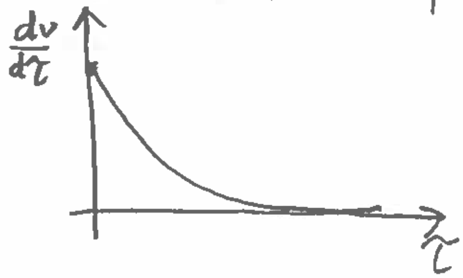
So the plot of $\frac{du}{d\tau}$ versus τ is then:



j.) Show that if $b>1$, then $t_{peak}=0$.

Solution:

If $b>1$ then the plot of $\frac{dv}{d\tau}$ is given by:



k.) Give a biological interpretation of the constant b.

Solution:

The constant b is given by $b = 1/x_0K$. This constant measures the ratio of the the death rate to the infection rate per number of initial healthy population.