Homework 3

Mathematical Modeling

Due: September 24, 2018

1 Problems for Everybody

1. Consider the logistic equation:
$$\dot{N} = rN\left(1 - \frac{N}{\kappa}\right),$$
where $r, \kappa > 0$ and $N(0) = N_0$.
   (a) The system has three dimensional parameters r, K, and N_0. Find the dimensions of these parameters.
   (b) Show that the system can be rewritten in the dimensionless form
   $$\frac{dx}{d\tau} = x(1-x), \text{ and } x(0) = x_0$$
   for appropriate choices of the dimensionless variables x, x_0 and τ.
   (c) Find a different nondimensionalization in terms of variables u and τ, where u is chosen such that the initial condition is always $u_0 = 1$.
2. Consider the following system:
$$\dot{x} = h + rx - x^2.$$
   (a) Plot the bifurcation diagram for $h < 0$, $h = 0$, and $h > 0$.
   (b) Sketch the regions in the (r, h) plane that correspond to qualitatively different phase portraits and identify the bifurcations that occur on the boundaries of those regions.
3. Find the conditions under which it is valid to approximate the equation $mL^2\ddot{\theta} + b\dot{\theta} + mgL\sin(\theta) = \Gamma$ by its overdamped limit:
$$b\dot{\theta} + mgL\sin(\theta) = \Gamma.$$
Here $m, L, g, \Gamma > 0$ are constants.

2 Problems for MST 651 students only. Students in MST 351 can complete these problems for extra credit

1. Zebra stripes and butterfly wing patterns are two of the most spectacular examples of biological pattern formation. Explaining the development of these patterns is one of the outstanding problems of biology. One ingredient in a model of pattern formation is to understand a simple example of a biochemical switch, in which a gene G is activated by a biochemical signal substance S. For example, a gene may normally be inactive but can be "switched on" to produce a pigment or other gene product when the concentration of S exceeds a certain threshold. Let $g(t)$ denote the concentration of the gene product, and assume that the concentration s_0 of S is fixed. The model is:
$$\dot{g} = k_1 s_0 - k_2 g + \frac{k_3 g^2}{k_4^2 + g^2}$$

where the k's are positive constants. The production of g is stimulated by s_0 at a rate k_1, and by an *autocatalytic* or positive feedback process (the nonlinear term). There is also a linear degradation of g at rate k_2.

(a) Show that the system can be put in the dimensionless form:

$$\frac{dx}{d\tau} = s - rx + \frac{x^2}{1+x^2}$$

where $r > 0$ and $s \geq 0$ are dimensionless groups.

(b) Show that if $s = 0$, there are two positive fixed points if $r < r_c$, where r_c is to be determined.

(c) Assume that initially there is no gene product, i.e., $g(0) = 0$, and suppose s is slowly increased from 0 (the activating signal is turned on); what happens to $g(t)$? What happens if s then goes back to zero? Does the gene turn off again?

Problems for Everybody

#1.

Consider the logistic equation:

$$\dot{N} = rN\left(1-\frac{N}{K}\right),$$

where $r, K > 0$ and $N(0) = N_0$.

a.) The system has three dimensional parameters r, K and N_0. Find the dimensions of these parameters.

Solution:

Calculating, it follows that

$$[r] = \frac{1}{\text{time}}, \quad [K] = \text{pop.}, \quad [N_0] = \text{pop.}$$

b.) Show that the system can be rewritten in the dimensionless form:

$$\frac{dx}{d\tau} = x(1-x), \quad x(0) = x_0$$

Solution:

Let $x = N/K$ and $\tau = rt$. Then,

$$\frac{dN}{dt} = Kr\frac{dx}{d\tau}$$

$$\Rightarrow \frac{dx}{d\tau} = x(1-x), \quad x_0 = \frac{N_0}{K}.$$

c.) Find a different nondimensionalization for which $u_0 = 1$.

Solution:

Let $u = \frac{N}{N_0}, \tau = rt$

$$\Rightarrow \frac{dN}{d\tau} = u\left(1 - \frac{N_0 u}{K}\right), \quad u_0 = 1.$$

#2.

Consider the following system:

$$\dot{x} = h + rx - x^2$$

a.) Plot the bifurcation diagram for $h<0$, $h=0$, and $h>0$.

Solution:

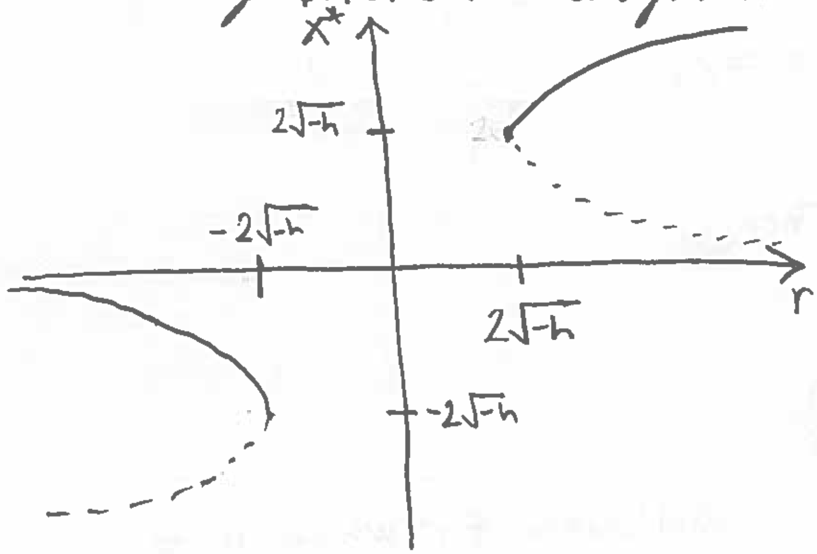
The fixed points if they exist satisfy

$$-x^2 - rx - h = 0$$

$$\Rightarrow x^* = \frac{r \pm \sqrt{r^2+4h}}{2}$$

Case 1:

If $h<0$ then the fixed points exist if $r^2 \geq -4h$. This yields the following bifurcation diagram:

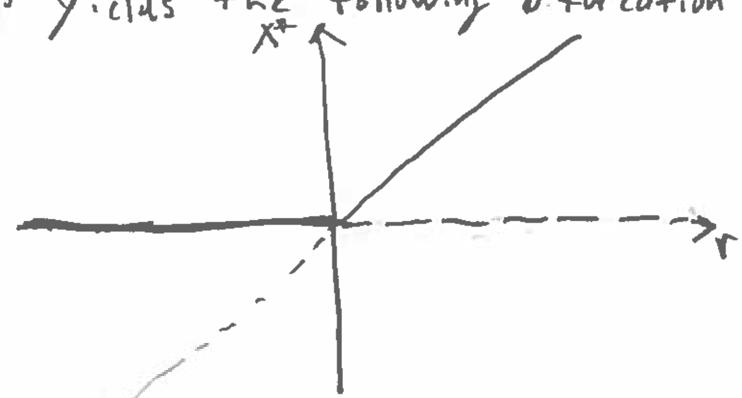


Case 2:

If $h=0$ the fixed points are given by

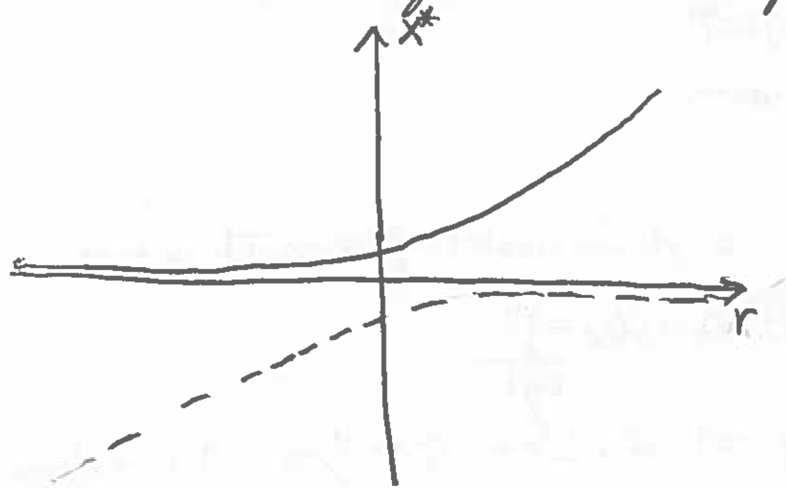
$$x^* = 0, r.$$

This yields the following bifurcation diagram:



Case 3:

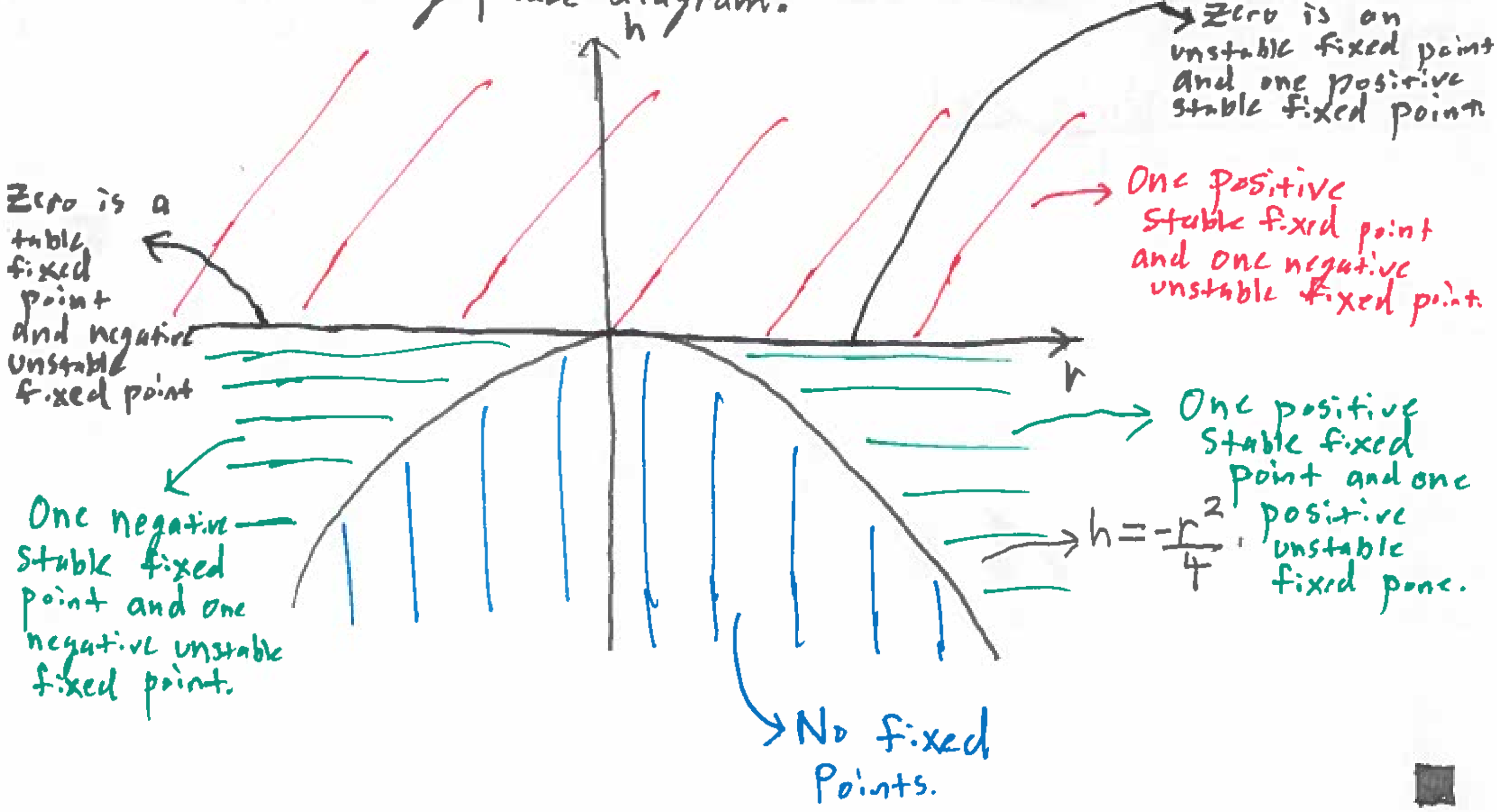
If $h>0$ the fixed points always exists and there are no bifurcations. The resulting bifurcation diagram is given by:



b.) Sketch the regions in the (r,h) plane that correspond to qualitatively different phase portraits and identify the bifurcations that occur on the boundaries of those regions.

Solution:

For $h \leq 0$ the bifurcations occur along the curve $h = -r^2/4$. This yields the following phase diagram:



#3.

Find the conditions under which it is valid to approximate the equation $mL^2\ddot{\theta}+b\dot{\theta}+mgL\sin(\theta)=\Gamma$ by its overdamped limit:

$$b\dot{\theta}+mgL\sin(\theta)=\Gamma.$$

Here $m, L, g, \Gamma > 0$ are constants.

Solution:

It is clear that mgL is a dimensionless group. Therefore,

$$\frac{mL^2}{mgL}\ddot{\theta}+\frac{b}{mgL}\dot{\theta}+\sin(\theta)=\frac{\Gamma}{mgL}$$

is a dimensionless equation. Let $\tau = t/T_{sc}$. It follows that

$$\frac{L}{gT_{sc}^2}\frac{d^2\theta}{d\tau^2}+\frac{b}{mgLT_{sc}}\frac{d\theta}{d\tau}+\sin\theta=\frac{\Gamma}{mgL}$$

We set $T_{sc}=\frac{mgL}{b}$ to set the coefficient of friction to unity.

$$\Longrightarrow \frac{b^2m^2g}{L^2}\frac{d^2\theta}{d\tau^2}+\frac{d\theta}{d\tau}+\sin\theta=\frac{\Gamma}{mgL}$$

Consequently, the condition that we can make the overdamped approximation is:

$$\frac{b^2m^2g}{L} \ll 1.$$

Graduate Problems

#1.

The model for gene concentration is given by:

$$\dot{g} = k_1 S_0 - k_2 g + \frac{k_3 g^2}{k_4^2 + g^2}.$$

a.) Show that the system can be put in dimensionless form:

$$\frac{dx}{d\tau} = s - rx + \frac{x^2}{1+x^2}.$$

Solution:

Factoring it follows that:

$$\dot{g} = k_1 S_0 - k_2 g + \frac{k_3}{k_4^2} \frac{g^2}{(1 + g^2/k_4^2)}.$$

Let $x = g/k_4$. Therefore,

$$k_4 \dot{x} = k_1 S_0 - k_2 k_4 x + \frac{k_3}{k_4} \frac{k_4^2 x^2}{(1+x^2)}$$

$$\Rightarrow \dot{x} = \frac{k_1 S_0}{k_4} - k_2 x + k_3 \frac{x^2}{(1+x^2)}$$

Let $\tau = k_3 t$. It follows that

$$\frac{dx}{d\tau} = s - rx + \frac{x^2}{1+x^2},$$

where $s = \frac{k_1 S_0}{k_3 k_4}$, $r = \frac{k_2}{k_3}$.

b.) Show that if $s=0$, there are two fixed points if $r<r_c$, where r_c is to be determined.

Solution:

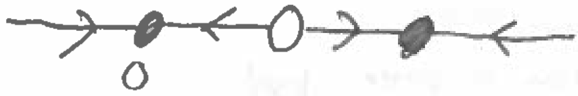
If $s=0$ then fixed points satisfy:

$$rx(1+x^2) - x^2 = 0$$

$$\Rightarrow x(r(1+x^2) - x) = 0$$

$$\Rightarrow x = 0, \quad x = \frac{1 \pm \sqrt{1-4r^2}}{2r}$$

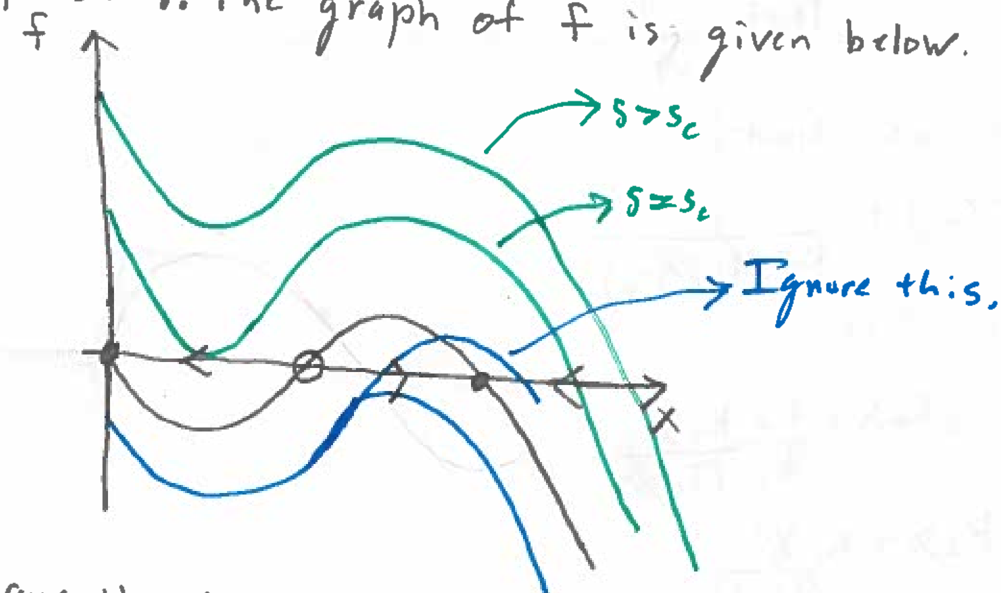
Consequently, if $r < \frac{1}{2}$ there are two positive fixed points. The phase portrait is given by:



C.) What happens as s is raised.

Solution:

Let $f(x) = -rx + \frac{x^2}{1+x^2}$. The effect of s is to vertically shift the graph of f. The graph of f is given below.



Consequently, there exists s_c at which a saddle node bifurcation. Post bifurcation the system undergoes hysteresis. Therefore, if s is lowered back to 0 the system instead remains at the other positive stable fixed point.