Homework 4

Mathematical Modeling

Due: October 3, 2018

1 Problems for Everybody

1. In this problem we study 2×2 systems of linear ODEs:

$$\dot{y} = Ay, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Classify the origin as a a stable/unstable spiral, node, or saddle, and sketch the phase portrait for each of the following cases:

$$A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} -1 & -2 \\ 2 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}.$$

2. Show that

$$x_1(t) = e^{\alpha t} \begin{bmatrix} \cos(\beta t) \\ -\sin(\beta t) \end{bmatrix} \text{ and } x_2(t) = \begin{bmatrix} \sin(\beta t) \\ \cos(\beta t) \end{bmatrix}$$

are two solutions of the linear differential equation:

$$\dot{x} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} x.$$

The principle of linear superposition ensure that all solutions can be written as a linear combination of x_1 and x_2:

$$x(t) = c_1 x_1(t) + c_2 x_2(t).$$

Write $x(t)$ in the following form:

$$x(t) = ae^{\alpha}(t) \begin{bmatrix} \cos(\beta t + \phi) \\ -\sin(\beta t + \phi) \end{bmatrix},$$

with $a = \sqrt{c_1^2 + c_2^2}$. The parameter ϕ is called the phase. Find an expression for the phase depending on c_1 and c_2.

3. For a 2×2 matrix A prove that the eigenvalues $\lambda_{1,2}$ of A satisfy:

$$\lambda_{1,2} = \frac{\mathrm{Tr}(A)}{2} \pm \frac{1}{2}\sqrt{(\mathrm{Tr}(A))^2 - 4\det(A)}.$$

Hint: You can use the fact that $\mathrm{Tr}(A) = \lambda_1 + \lambda_2$ and $\det(A) = \lambda_1 \lambda_2$.

Problems for Everybody

#1.

Classify the origin as stable/unstable spiral, node, or saddle, and sketch the phase portrait for the system $\dot{\vec{x}} = A\vec{x}$.

$A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$:

The eigenvalues satisfy:

$$\lambda_1 \lambda_2 = -4$$
$$\lambda_1 + \lambda_2 = 0$$
$$\Rightarrow \lambda_1 = 2, \lambda_2 = -2.$$

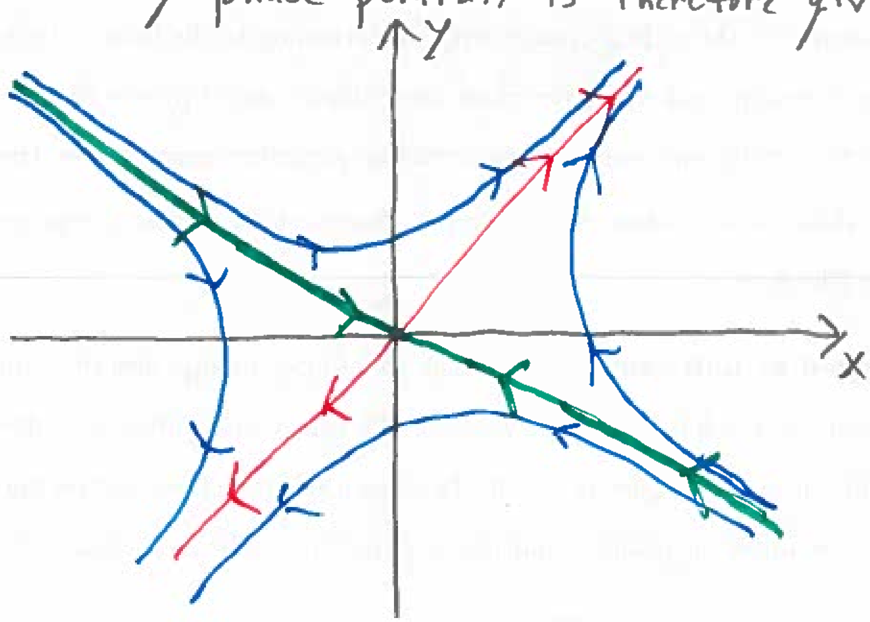
The eigenvectors are in the nullspace of the following matrices:

$$A_{\lambda_1} = \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix} \text{ and } A_{\lambda_2} = \begin{bmatrix} 3 & 1 \\ 3 & -1 \end{bmatrix}$$

Consequently, eigenvectors $\vec{v}_1, \vec{v}_2$ corresponding to λ_1 and λ_2 are given by:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}.$$

The resulting phase portrait is therefore given by:



Saddle Node

$$A=\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}.$$

The eigenvalues satisfy:

$$\lambda_1\lambda_2=4$$
$$\lambda_1+\lambda_2=5$$
$$\Rightarrow \lambda_1+\frac{4}{\lambda_1}=5$$
$$\Rightarrow \lambda_1^2-5\lambda_1+4=0$$
$$\Rightarrow (\lambda_1-4)(\lambda_1-1)=0$$
$$\Rightarrow \lambda_1=4,\ \lambda_2=1.$$

The eigenvectors are in the nullspace of the following matrices

$$A_{\lambda_1}=\begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} \text{ and } A_{\lambda_2}=\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

Consequently, eigenvectors corresponding to λ_1 and λ_2 are given by:

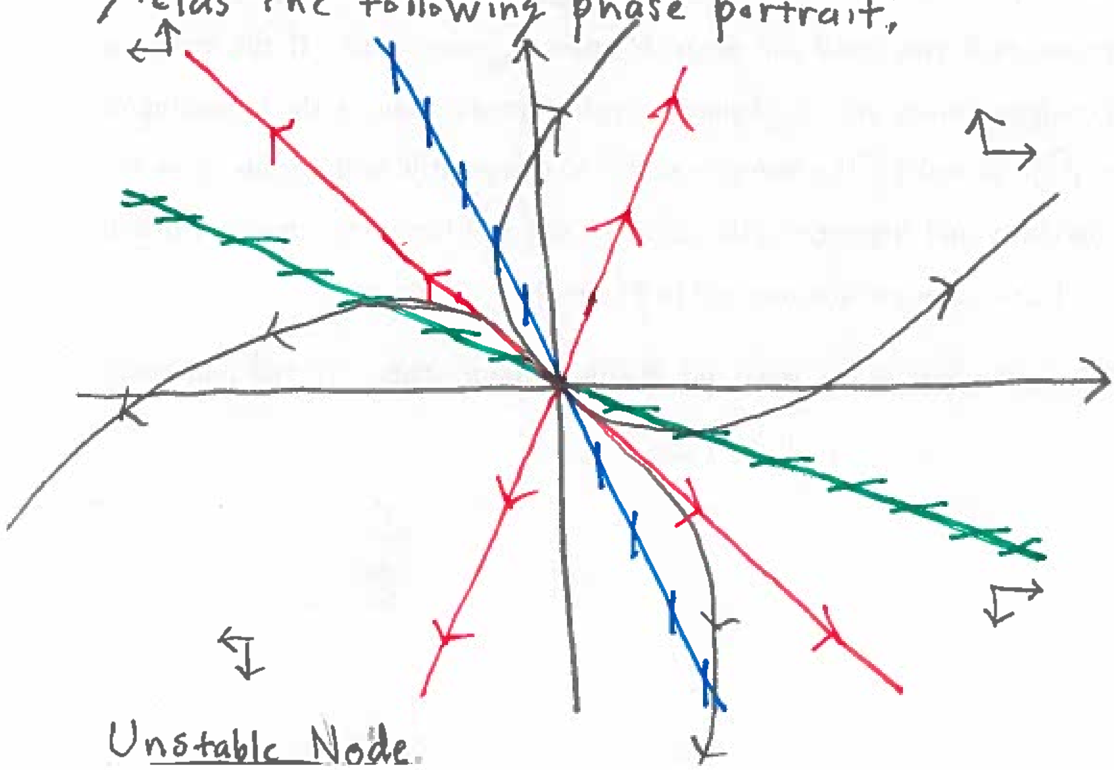
$$\vec{v}_1=\begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \text{ and } \vec{v}_2=\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

The nullclines for this system are given by:

N1: $y=-2x$ $(\dot{x}=0)$

N2: $y=-\frac{2}{3}x$ $(\dot{y}=0)$.

This yields the following phase portrait:



Unstable Node.

$$A = \begin{bmatrix} -1 & -2 \\ 2 & -1 \end{bmatrix}:$$

The eigenvalues satisfy:

$$\lambda_1 \lambda_2 = 5$$

$$\lambda_1 + \lambda_2 = -2$$

$$\Rightarrow \lambda_1 + \frac{5}{\lambda_1} = -2$$

$$\Rightarrow \lambda_1^2 + 2\lambda_1 + 5 = 0$$

$$\Rightarrow \lambda_{1,2} = \frac{-2 \pm \sqrt{4-20}}{2}$$

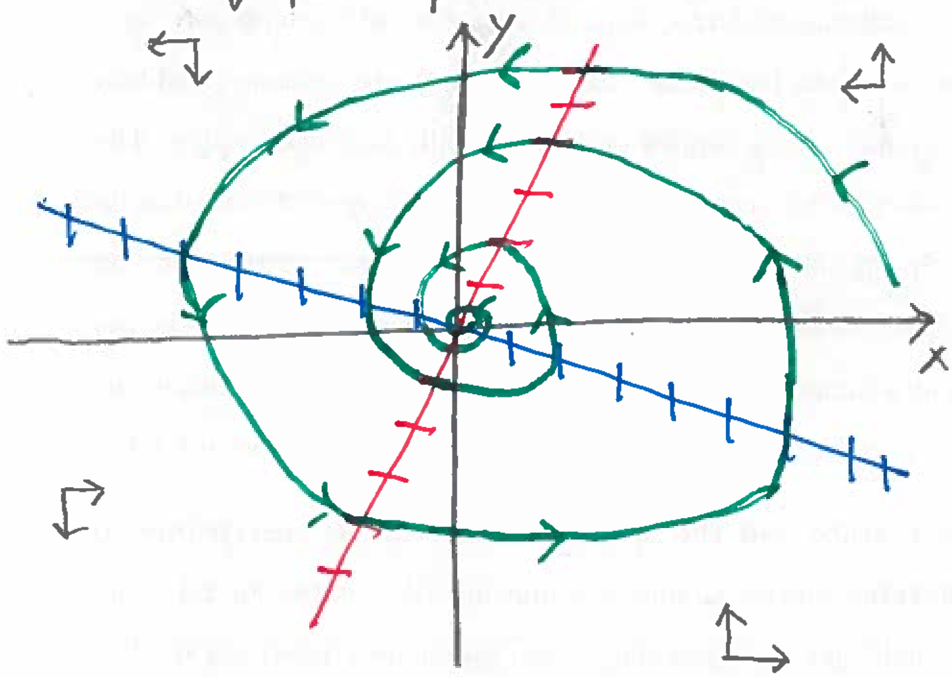
$$\Rightarrow \lambda_{1,2} = -1 \pm 2i.$$

Therefore, the origin is a stable spiral. The nullclines are given by:

<u>N1</u>: $y = -\frac{1}{2}x$ $(\dot{x} = 0)$

<u>N2</u>: $y = 2x$ $(\dot{y} = 0)$

The resulting phase portrait is therefore:



<u>Stable Spiral</u>

$$\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}:$$

The eigenvalues satisfy:

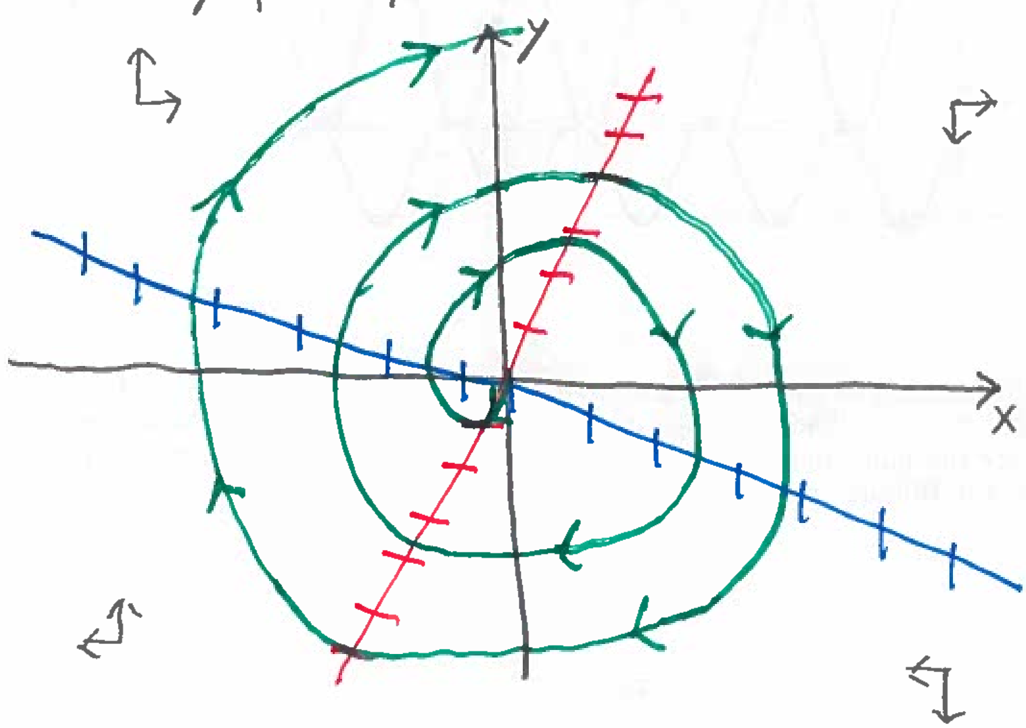
$$\lambda_1 \lambda_2 = 5$$
$$\lambda_1 + \lambda_2 = 2$$
$$\Rightarrow \lambda_1 + \frac{5}{\lambda_1} = 2$$
$$\Rightarrow \lambda_1^2 - 2\lambda_1 + 5 = 0$$
$$\Rightarrow \lambda_{1,2} = \frac{2 \pm \sqrt{4-20}}{2}$$
$$\Rightarrow \lambda_{1,2} = 1 \pm 2i$$

Therefore, the origin is an unstable spiral. The nullclines are given by:

N1: $y = -\frac{1}{2}x$ $(\dot{x} = 0)$

N2: $y = 2x$ $(\dot{y} = 0)$

The resulting phase portrait is therefore:



Unstable Spiral

#2.

Show that solutions of the differential equation

$$\dot{X} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} X$$

are of the form

$$X(t) = a e^{\alpha t} \begin{bmatrix} \cos(\beta t + \phi) \\ -\sin(\beta t + \phi) \end{bmatrix}.$$

Solution:

By direct compution two linearly independent solutions are given by:

$$X_1(t) = e^{\alpha t} \begin{bmatrix} \cos(\beta t) \\ -\sin(\beta t) \end{bmatrix} \text{ and } X_2(t) = e^{\alpha t} \begin{bmatrix} \sin(\beta t) \\ \cos(\beta t) \end{bmatrix}$$

A generic solution can thus be expressed as

$$X(t) = c_1 X_1(t) + c_2 X_2(t)$$

$$\Rightarrow X(t) = e^{\alpha t}\left(c_1 \begin{bmatrix} \cos(\beta t) \\ -\sin(\beta t) \end{bmatrix} + c_2 \begin{bmatrix} \sin(\beta t) \\ \cos(\beta t) \end{bmatrix} \right)$$

$$= \sqrt{c_1^2 + c_2^2}\, e^{\alpha t} \left(\frac{c_1}{\sqrt{c_1^2+c_2^2}} \begin{bmatrix} \cos(\beta t) \\ -\sin(\beta t) \end{bmatrix} + \frac{c_2}{\sqrt{c_1^2+c_2^2}} \begin{bmatrix} \sin(\beta t) \\ \cos(\beta t) \end{bmatrix} \right)$$

$$= \sqrt{c_1^2 + c_2^2}\, e^{\alpha t} \begin{bmatrix} \cos(\beta t + \phi) \\ -\sin(\beta t + \phi) \end{bmatrix},$$

Where

$$\cos(\phi) = \frac{c_1}{\sqrt{c_1^2 + c_2^2}} \text{ and } \sin(\phi) = -\frac{c_2}{\sqrt{c_1^2 + c_2^2}}$$

$$\Rightarrow \phi = \tan^{-1}\left(\frac{-c_2}{c_1}\right).$$

■

#3.

For a 2×2 matrix A prove that the eigenvalues $\lambda_{1,2}$ of A satisfy:

$$\lambda_{1,2}=\frac{Tr(A)}{2}\pm\frac{1}{2}\sqrt{(Tr(A))^2-4\det(A)}.$$

Solution:

Since $\lambda_1+\lambda_2=Tr(A)$ and $\det(A)=\lambda_1\lambda_2$ it follows that

$$\lambda_1+\frac{\det(A)}{\lambda_1}=Tr(A)$$

$$\Rightarrow \lambda_1^2-Tr(A)\lambda_1+\det(A)=0$$

$$\Rightarrow \lambda_{1,2}=\frac{Tr(A)\pm\sqrt{(Tr(A))^2-4\det(A)}}{2}$$

#4.

Analyze the following system:

$$\dot{x}=y$$
$$\dot{y}=x$$

Solution:

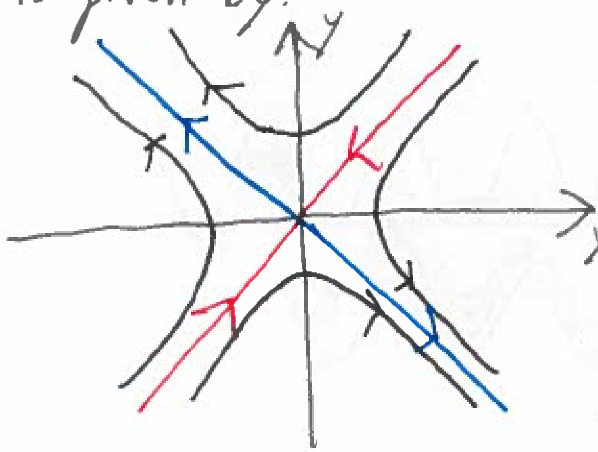
Calculating it follows that

$$\frac{dy}{dx}=\frac{x}{y}$$

$$\Rightarrow \int y\,dy=\int x\,dx$$

$$\Rightarrow y^2-x^2=C,$$

where C is a constant. Consequently, the solution curves are hyperbolas if $C\neq0$ and the lines $y=\pm x$ if $C=0$. The resulting phase portrait is given by.



#5.

The motion of a damped harmonic oscillator is described by $m\ddot{x}+b\dot{x}+Kx=0$, where $m,b,K>0$ are constants.

a.) Rewrite the equation as a two-dimensional linear system.

Solution:

Let $v=\dot{x}$. The resulting system is then:

$$\dot{x}=v$$
$$\dot{v}=-\frac{b}{m}v-\frac{K}{m}x$$

b.) Classify the fixed point at the origin and sketch the phase portrait. How do your results relate to the standard notions of overdamped, critically damped, and underdamped vibrations?

Solution:

Let $\alpha = b/m$ and $\beta = K/m$. Then

$$\dot{x}=v \qquad \dot{v}=-\alpha v-\beta x \Rightarrow \begin{bmatrix}\dot{x}\\ \dot{v}\end{bmatrix}=\begin{bmatrix}0 & 1\\ -\beta & -\alpha\end{bmatrix}\begin{bmatrix}x\\ v\end{bmatrix}$$

The eigenvalues satisfy:

$$\lambda_1+\lambda_2=-\alpha$$
$$\lambda_1\lambda_2=\beta$$
$$\Rightarrow \lambda_1+\frac{\beta}{\lambda_1}=-\alpha$$
$$\Rightarrow \lambda_1^2+\lambda_1\alpha+\beta=0$$
$$\Rightarrow \lambda_1=\frac{-\alpha\pm\sqrt{\alpha^2-4\beta}}{2}$$

The nullclines for this system are given by

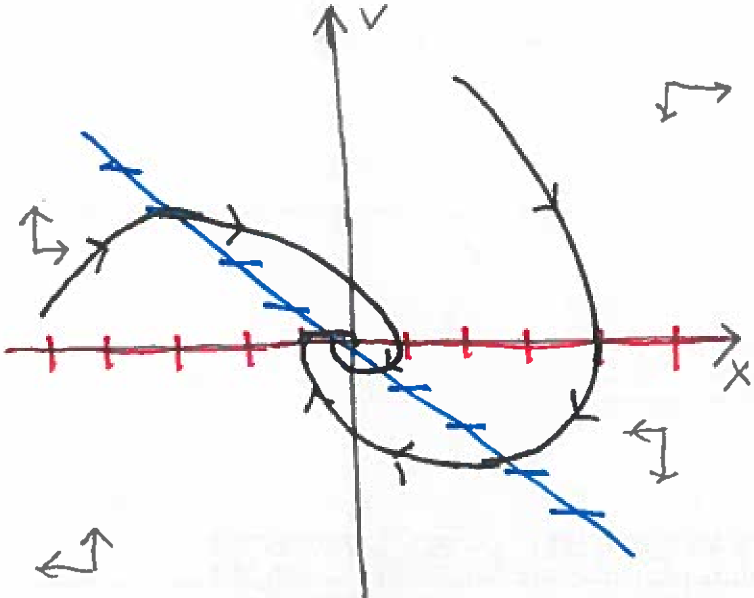
$\underline{N1}$: $v=0$ $(\dot{x}=0)$

$\underline{N2}$: $v=-\beta/\alpha\, x$ $(\dot{y}=0)$

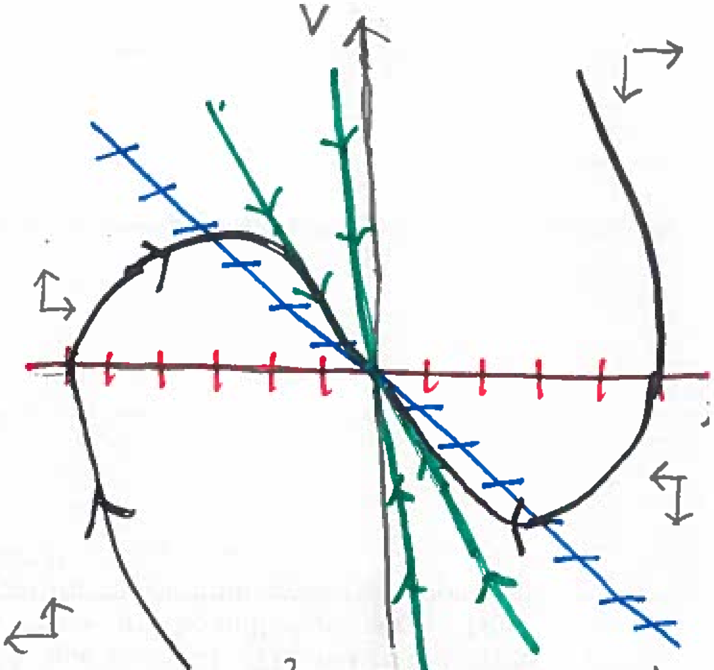
The eigenvectors are given by:

$$\vec{v}_1=\left[-\frac{\alpha+\sqrt{\alpha^2-4\beta}}{2\beta},1\right]^T,\quad \vec{v}_2=\left[-\frac{\alpha-\sqrt{\alpha^2-4\beta}}{2\beta},1\right]^T.$$

The phase portraits are therefore given by.



$\alpha^2 < 4\beta$ (underdamped)



$\alpha^2 > 4\beta$ (overdamped)

MST: 651 Problem

#1.

For each of the systems decide whether the origin is attracting, Liapunov stable, asymptotically stable, or none of the above.

a) $\dot{x} = y$ and $\dot{y} = -4$.

Solution:

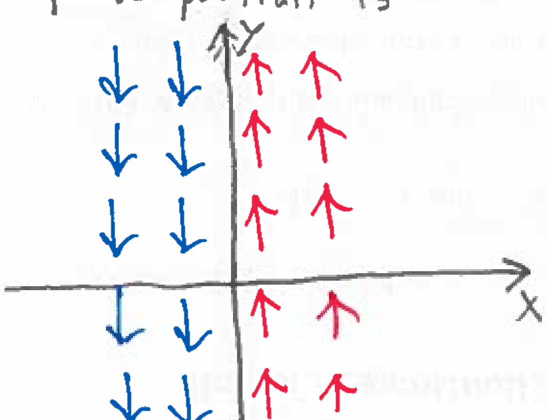
$$A = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \Rightarrow \lambda_{1,2} = \pm 2i.$$

Liapunov stable.

b.) $\dot{x} = 0$, $\dot{y} = x$

Solution:

The phase portrait is



This is neither.

c.) $\dot{x} = -x$, $\dot{y} = -5y$

Solution:

The flow is inward towards $(0,0)$ and thus $(0,0)$ is stable.

d.) $\dot{x} = 2y$, $\dot{y} = x$

Solution:

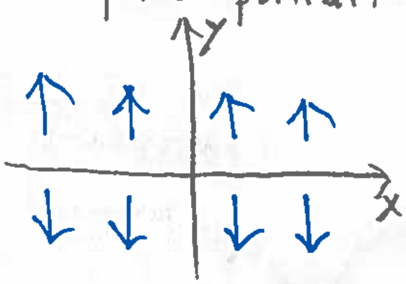
$$A = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \Rightarrow \lambda_{1,2} = \pm\sqrt{2}.$$

Consequently is neither Lyapunov stable or attracting.

e.) $\dot{x} = 0$, $\dot{y} = y$

Solution:

The phase portrait is given by



Clearly, this system is neither attracting or Lyapunov stable.

f.) $\dot{x} = x$, $\dot{y} = y$

Solution:

The flow is outward from $(0,0)$ and consequently $(0,0)$ is neither attracting or Lyapunov stable.