Homework 5

Mathematical Modeling

Due: October 10, 2018

1 Problems for Everybody

1. For the following systems, find the fixed points and analyze their stability, sketch the nullclines and a plausible phase portrait.

   (a) $\dot{x} = x - x^3$ and $\dot{y} = -y$.

   (b) $\dot{x} = y$ and $\dot{y} = x(1+y) - 1$.

   (c) $\dot{x} = \sin(y)$ and $\dot{y} = x - x^3$.

   (d) $\dot{x} = xy - 1$ and $\dot{y} = x - y^3$.

2. In this problem we study the competing species model:

$$\begin{cases} \dot{x} = ax + bxy \\ \dot{y} = cy + dxy \end{cases},$$

   where $a, b, c, d \in \mathbb{R}$. In the following cases, sketch the phase portrait, analyze the stability of any fixed points, and give a biological interpretation for each case.

   (a) $a, b, c, d > 0$.

   (b) $a, b, c > 0$ and $d < 0$.

   (c) $a, b, d > 0$ and $c < 0$.

   (d) $a, b > 0$ and $c, d < 0$.

   (e) $b, c > 0$ and $a, d < 0$.

   (f) $a, c > 0$ and $b, d < 0$.

   (g) $b, d > 0$ and $a, c < 0$.

   (h) $b > 0$ and $a, c, d < 0$.

   (i) $a > 0$ and $b, c, d < 0$.

   (j) $a, b, c, d < 0$.

3. Consider the following model for the interaction of the population of deer N_1 and rabbits N_2:

$$\begin{cases} \dot{N}_1 = r_1 N_1 \left(1 - \frac{N_1}{\kappa_1}\right) - \alpha N_1 N_2 \\ \dot{N}_2 = r_2 N_2 \left(1 - \frac{N_1}{\kappa_2}\right) - \beta N_1 N_2 \end{cases},$$

   where $r_1, r_2, \kappa_1, \kappa_2, \alpha, \beta$ are constants.

   (a) Give biological interpretations of each of the parameters.

   (b) Nondimensionalize this system. There are many ways to do this. You should do the most natural one that makes sense biologically and makes the problem as simple as possible.

(c) Classify the fixed points for this system and sketch the phase portrait. Be sure to show all the different cases that can occur, depending on the relative sizes of the parameters.

4. In an open-access fishery, fisherman are free to come and go as they please. The fishing effort E is determined by the opportunity to make a profit. Let $c > 0$ be the cost of operation, and $p > 0$ the price the fisherman get for their catch H.

   (a) Explain why $P = pH - cE$, where $H = qEN$ for some constant $q > 0$, is a realistic model for the total profit of the fisherman.

   (b) Explain why
   $$\dot{E} = aP, \tag{1}$$
   where $a > 0$, is a realistic model of effort.

   (c) Explain why
   $$\dot{N} = rN\left(1 - \frac{N}{\kappa}\right) - H, \tag{2}$$
   where $r, \kappa > 0$ are constants, is a realistic model of fish population.

   (d) Nondimensionalize the coupled system defined by equations (1) and (2).

   (e) Sketch phase portraits and analyze the stability of fixed points for your dimensionless system.

5. Consider the system $\dot{x} = xy$ and $\dot{y} = x^2 - y$.

   (a) Show the linearization predicts that the origin is a non-isolated fixed point.

   (b) Show that origin is in fact an isolated fixed point.

   (c) Sketch the phase portrait for this system. Is the origin repelling, attracting, a saddle or what?

2 Problems for MST 651 students only. Students in MST 351 can complete these problems for extra credit

1. Consider the system $\dot{x} = y^3 - 4x$ and $\dot{y} = y^3 - y - 3x$.

   (a) Find all the fixed points and classify them.

   (b) Prove that the line $y = x$ is invariant.

   (c) Prove that $|x(t) - y(t)| \to 0$ as $t \to \infty$.

   (d) Sketch the phase portrait.