Homework 5

Mathematical Modeling

Due: October 10, 2018

1 Problems for Everybody

1. For the following systems, find the fixed points and analyze their stability, sketch the nullclines and a plausible phase portrait.

   (a) $\dot{x} = x - x^3$ and $\dot{y} = -y$.

   (b) $\dot{x} = y$ and $\dot{y} = x(1 + y) - 1$.

   (c) $\dot{x} = \sin(y)$ and $\dot{y} = x - x^3$.

   (d) $\dot{x} = xy - 1$ and $\dot{y} = x - y^3$.

2. In this problem we study the competing species model:

$$\begin{cases} \dot{x} = ax + bxy \\ \dot{y} = cy + dxy \end{cases},$$

   where $a, b, c, d \in \mathbb{R}$. In the following cases, sketch the phase portrait, analyze the stability of any fixed points, and give a biological interpretation for each case.

   (a) $a, b, c, d > 0$.

   (b) $a, b, c > 0$ and $d < 0$.

   (c) $a, b, d > 0$ and $c < 0$.

   (d) $a, b > 0$ and $c, d < 0$.

   (e) $b, c > 0$ and $a, d < 0$.

   (f) $a, c > 0$ and $b, d < 0$.

   (g) $b, d > 0$ and $a, c < 0$.

   (h) $b > 0$ and $a, c, d < 0$.

   (i) $a > 0$ and $b, c, d < 0$.

   (j) $a, b, c, d < 0$.

3. Consider the following model for the interaction of the population of deer N_1 and rabbits N_2:

$$\begin{cases} \dot{N}_1 = r_1 N_1 \left(1 - \frac{N_1}{\kappa_1}\right) - \alpha N_1 N_2 \\ \dot{N}_2 = r_2 N_2 \left(1 - \frac{N_1}{\kappa_2}\right) - \beta N_1 N_2 \end{cases},$$

   where $r_1, r_2, \kappa_1, \kappa_2, \alpha, \beta$ are constants.

   (a) Give biological interpretations of each of the parameters.

   (b) Nondimensionalize this system. There are many ways to do this. You should do the most natural one that makes sense biologically and makes the problem as simple as possible.

(c) Classify the fixed points for this system and sketch the phase portrait. Be sure to show all the different cases that can occur, depending on the relative sizes of the parameters.

4. In an open-access fishery, fisherman are free to come and go as they please. The fishing effort E is determined by the opportunity to make a profit. Let $c > 0$ be the cost of operation, and $p > 0$ the price the fisherman get for their catch H.

   (a) Explain why $P = pH - cE$, where $H = qEN$ for some constant $q > 0$, is a realistic model for the total profit of the fisherman.

   (b) Explain why
   $$\dot{E} = aP, \tag{1}$$
   where $a > 0$, is a realistic model of effort.

   (c) Explain why
   $$\dot{N} = rN\left(1 - \frac{N}{\kappa}\right) - H, \tag{2}$$
   where $r, \kappa > 0$ are constants, is a realistic model of fish population.

   (d) Nondimensionalize the coupled system defined by equations (1) and (2).

   (e) Sketch phase portraits and analyze the stability of fixed points for your dimensionless system.

5. Consider the system $\dot{x} = xy$ and $\dot{y} = x^2 - y$.

   (a) Show the linearization predicts that the origin is a non-isolated fixed point.

   (b) Show that origin is in fact an isolated fixed point.

   (c) Sketch the phase portrait for this system. Is the origin repelling, attracting, a saddle or what?

2 Problems for MST 651 students only. Students in MST 351 can complete these problems for extra credit

1. Consider the system $\dot{x} = y^3 - 4x$ and $\dot{y} = y^3 - y - 3x$.

   (a) Find all the fixed points and classify them.

   (b) Prove that the line $y = x$ is invariant.

   (c) Prove that $|x(t) - y(t)| \to 0$ as $t \to \infty$.

   (d) Sketch the phase portrait.

Problems for Everybody

#1.

For the following systems, find the fixed points and analyze their stability, sketch the nullclines and a plausible phase portrait.

a.) $\dot{x} = x - x^3$, $\dot{y} = -y$.

Solution:

The nullclines are given by

N1: $x=0$ $(\dot{x}=0)$

N2: $x=1$ $(\dot{x}=0)$

N3: $x=-1$ $(\dot{x}=0)$

N4: $y=0$ $(\dot{y}=0)$

Consequently, the fixed points are given by:

$(-1,0)$, $(0,0)$, $(1,0)$.

The Jacobian is given by:

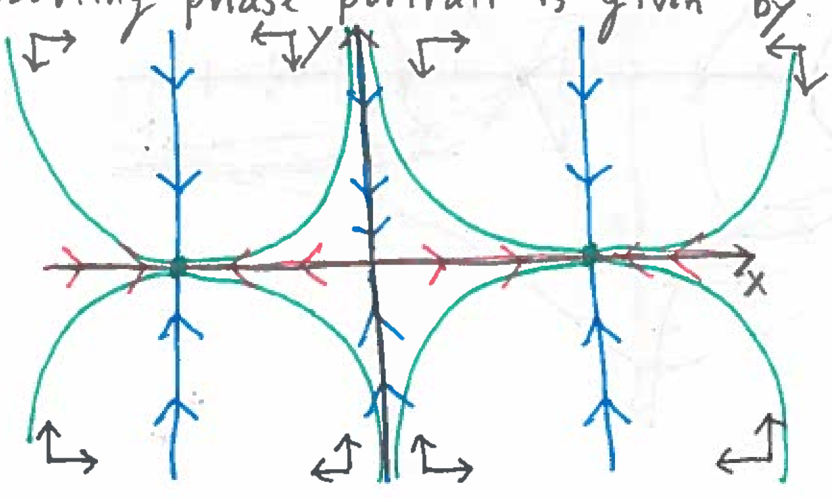
$$J(x,y) = \begin{bmatrix} 1-3x^2 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow J(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow (0,0) \text{ is saddle-node}$$

$$J(-1,0) = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow (-1,0) \text{ is a stable node}$$

$$J(1,0) = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow (1,0) \text{ is a stable node.}$$

The resulting phase portrait is given by:



b.) $\dot{x} = y$ and $\dot{y} = x(1+y) - 1$

Solution:

The nullclines are given by

N1: $y = 0$ $(\dot{x} = 0)$

N2: $y = \frac{1-x}{x}$ $(\dot{y} = 0)$

The only fixed point is $(1,0)$ and the Jacobian is given by:

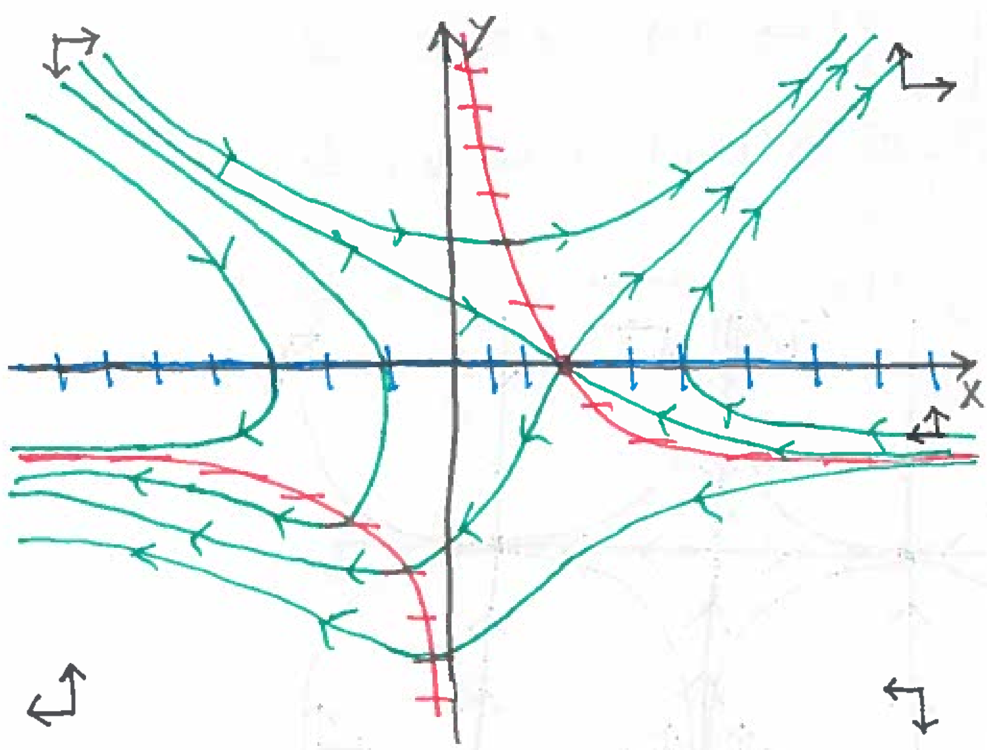
$$J(x,y) = \begin{bmatrix} 0 & 1 \\ 1+y & x \end{bmatrix}$$

$$\Rightarrow J(1,0) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

The eigenvalues satisfy

$$\lambda_1 \lambda_2 = -1$$
$$\lambda_1 + \lambda_2 = 1$$
$$\Rightarrow \lambda_1 - \frac{1}{\lambda_1} = 1$$
$$\Rightarrow \lambda_1^2 - \lambda_1 - 1 = 0$$
$$\Rightarrow \lambda_1 = \frac{1 \pm \sqrt{1+4}}{2}$$

Consequently, $(1,0)$ is a saddle. The resulting phase portrait is given by:



c.) $\dot{x} = \sin(y)$ and $\dot{y} = x - x^3$.

Solution:

N1: $\sin(y) = 0 \Rightarrow y = \pi n$. $(\dot{x}=0)$

N2: $x = 0$ $(\dot{y}=0)$

N3: $x = \pm 1$ $(\dot{y}=0)$

The fixed points are given by $(0, \pi n)$, $(\pm 1, \pi n)$. The Jacobian is given by:

$$J(x,y) = \begin{bmatrix} 0 & \cos(y) \\ 1-3x^2 & 0 \end{bmatrix}$$

$$\Rightarrow J(0,\pi n) = \begin{bmatrix} 0 & (-1)^n \\ 1 & 0 \end{bmatrix} \Rightarrow \lambda_1 \lambda_2 = (-1)^{n+1} \text{ and } \lambda_1 + \lambda_2 = 0$$

$$\Rightarrow -\lambda_1^2 = (-1)^{n+1}$$

$$\Rightarrow \lambda_{1,2} = \pm\sqrt{(-1)^n}$$

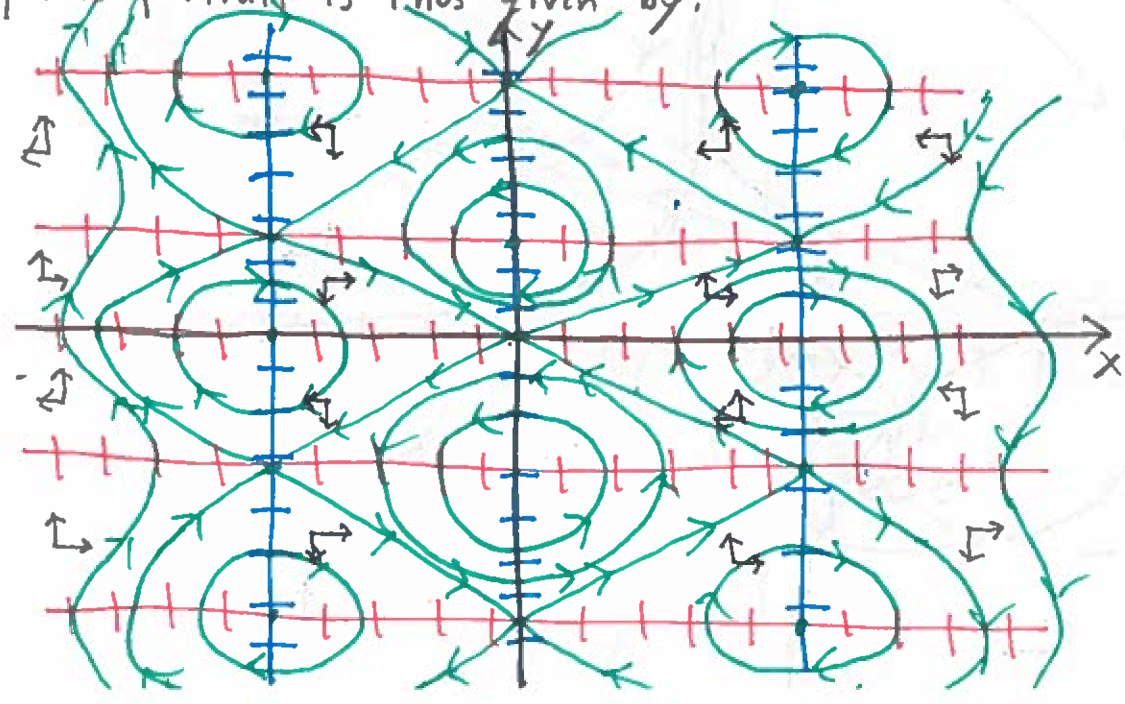
$$J(\pm 1, \pi n) = \begin{bmatrix} 0 & (-1)^n \\ -1 & 0 \end{bmatrix} \Rightarrow \lambda_{1,2} = \pm\sqrt{(-1)^{n+1}}$$

When $\lambda_{1,2}$ are imaginary, more work is needed to analyze the fixed point. To do this, note

$$\frac{dy}{dx} = \frac{x - x^3}{\sin(y)} \Rightarrow \frac{x^2}{2} - \frac{x^4}{4} + \cos(y) = C.$$

That is, the solution curves are contours of $F(x,y) = \frac{x^2}{2} - \frac{x^4}{4} + \cos(y)$.

The phase portrait is thus given by:



d.) $\dot{x} = xy - 1$ and $\dot{y} = x - y^3$.

Solution:

N1: $y = 1/x \; (\dot{x} = 0)$

N2: $y = x^{1/3} \; (\dot{y} = 0)$

The fixed points are $(-1,-1)$ and $(1,1)$. The Jacobian is given by

$$J(x,y) = \begin{bmatrix} y & x \\ 1 & -3y^2 \end{bmatrix}.$$

$$\Rightarrow J(-1,-1) = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix} \Rightarrow \lambda_1 \lambda_2 = 4 \quad \lambda_1 + \lambda_2 = -4$$

$$\Rightarrow \lambda_1 + \frac{4}{\lambda_1} = -4 \Rightarrow \lambda_1^2 + 4\lambda_1 + 4 = 0$$

$$\Rightarrow \lambda_{1,2} = -2 \quad \text{(stable node)}$$

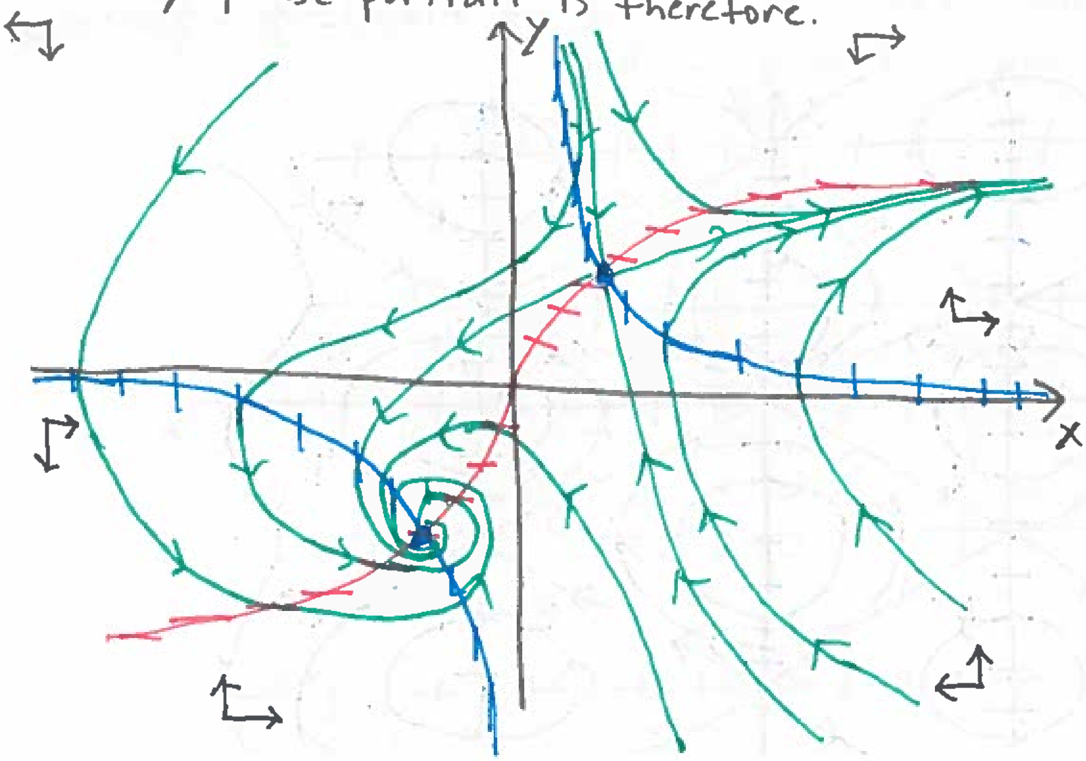
$$\Rightarrow J(1,1) = \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix} \Rightarrow \lambda_1 \lambda_2 = -4, \quad \lambda_1 + \lambda_2 = -2$$

$$\Rightarrow \lambda_1 - \frac{4}{\lambda_1} = -2$$

$$\Rightarrow \lambda_1^2 + 2\lambda_1 - 4 = 0$$

$$\Rightarrow \lambda_1 = \frac{-2 \pm \sqrt{4+16}}{2} \quad \text{(saddle-node)}$$

The resulting phase portrait is therefore.



#2

In this problem we study the competing species model:

$$\begin{cases} \dot{x} = ax + bxy \\ \dot{y} = cy + dxy \end{cases}$$

where $a, b, c, d \in \mathbb{R}$. In the following cases, sketch the phase portrait, analyze the stability of any fixed points, and give a biological interpretation for each case.

<u>Solution:</u>

We first analyze the system in full generality. The nullclines are given by:

<u>N1</u>: $x = 0$ $(\dot{x} = 0)$

<u>N2</u>: $y = 0$ $(\dot{y} = 0)$

<u>N3</u>: $y = -a/b$ $(\dot{x} = 0)$

<u>N4</u>: $x = -c/d$ $(\dot{y} = 0)$

Consequently, there are only two fixed points $(0,0)$ and $(-c/d, -a/b)$.

The Jacobian is given by:

$$J(x,y) = \begin{bmatrix} a+by & bx \\ dy & c+dx \end{bmatrix}.$$

$$\Rightarrow J(0,0) = \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}$$

There are three cases:

$a, c > 0 \Rightarrow (0,0)$ is an unstable node.

$a \cdot c < 0 \Rightarrow (0,0)$ is a saddle

$a, c < 0 \Rightarrow (0,0)$ is a stable node.

$$J(-c/d, -a/b) = \begin{bmatrix} 0 & -bc/d \\ -ad/b & 0 \end{bmatrix}$$

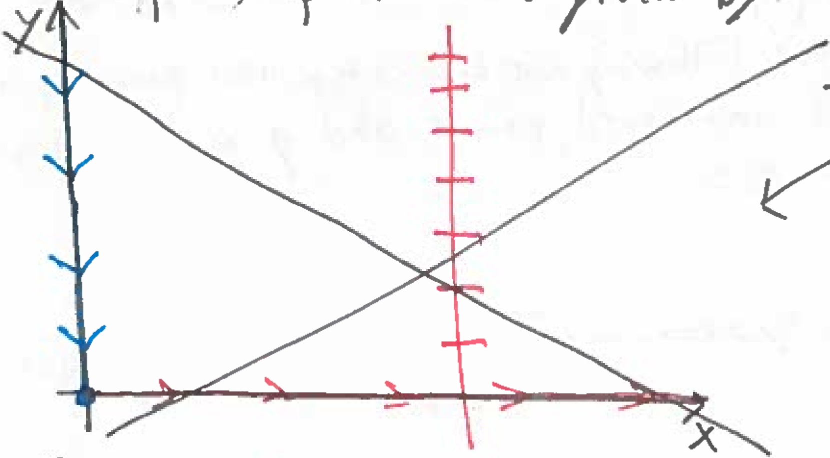
The eigenvalues are given by:

$$\lambda_{1,2} = \pm\sqrt{ac}$$

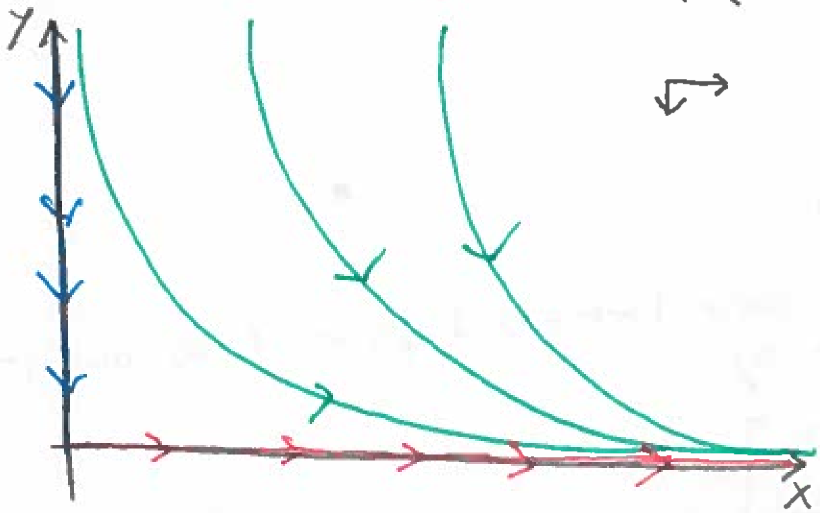
Consequently, when $(0,0)$ is a stable or unstable node then $(-c/d, -a/b)$ is a sadd d.) $a, b > 0$ and $c, d < 0$

Solution:

In this case $(0,0)$ is saddle and $(-c/d, -a/b)$ is not in the first quadrant. The phase portrait is given by:



This is wrong!

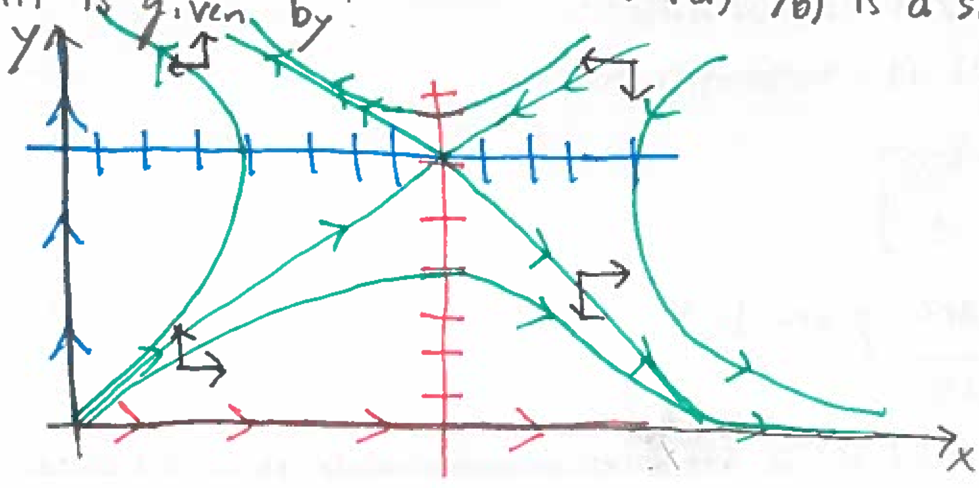


This system could model an ominivore (x) preying on a weaker predator (y). In this case species y goes extinct.

f.) $a, c > 0$ and $b, d < 0$

Solution:

In this case $(0,0)$ is unstable and $(-c/d, -a/b)$ is a saddle. The phase portrait is given by:



In this case both species are omnivores that prey on each other. Depending on initial conditions, one species goes extinct while the other grows exponentially.

#3

Analyze the following system:

$$\dot{N}_1 = r_1 N_1\left(1-\frac{N_1}{K_1}\right) - \alpha N_1 N_2$$

$$\dot{N}_2 = r_2 N_2\left(1-\frac{N_2}{K_2}\right) - \beta N_1 N_2$$

<u>Solution:</u>

Letting $x = N_1/K_1$, $y = N_2/K_2$, and $\tau = r_1 t$.

$$\Rightarrow \frac{dx}{d\tau} = x(1-x) - \gamma xy$$

$$\frac{dy}{d\tau} = \delta y(1-y) - \rho xy$$

where $\gamma, \delta, \rho > 0$. The nullclines are given by:

<u>N1</u>: $x = 0$ $(\dot{x}=0)$

<u>N2</u>: $y = \frac{1}{\gamma}(1-x)$ $(\dot{x}=0)$

<u>N3</u>: $y = 0$ $(\dot{y}=0)$

<u>N4</u>: $y = -\frac{\rho}{\delta}x + 1$ $(\dot{y}=0)$

The Jacobian is given by:

$$J(x,y) = \begin{bmatrix} 1-2x-\gamma y & -\gamma x \\ -\rho y & \delta - 2\delta y - \rho x \end{bmatrix}.$$

Three obvious fixed points are given by

$(0,0)$, $(0,1)$, $(1,0)$.

$$\Rightarrow J(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & \delta \end{bmatrix},\ J(0,1) = \begin{bmatrix} 1-\gamma & 0 \\ -\rho & -\delta \end{bmatrix},\ J(1,0) = \begin{bmatrix} -1 & -\gamma \\ 0 & \delta-\rho \end{bmatrix}$$

This generates the following cases:

$\underline{\gamma > 1,\ \rho > \delta}$

Both $(1,0)$ and $(0,1)$ are stable and $(0,0)$ is unstable.

$\underline{\gamma < 1,\ \rho < \delta}$

Both $(1,0)$ and $(0,1)$ are saddles and $(0,0)$ is unstable.

$\underline{\gamma < 1,\ \rho > \delta}$

$(0,1)$ is a saddle, $(1,0)$ is stable and $(0,0)$ is unstable.

$\underline{\gamma > 1,\ \rho < \delta}$

$(1,0)$ is a saddle, $(0,1)$ is stable and $(0,0)$ is unstable.

We now determine whether a fourth fixed point exists. This occurs if

$$\frac{1}{\gamma}(1-x) = -\frac{\rho}{\delta}x + 1$$

$$\Rightarrow 1 - x = -\frac{\rho\gamma}{\delta}x + \gamma$$

$$\Rightarrow 1 - \gamma = x\left(\frac{\delta - \rho\gamma}{\delta}\right)$$

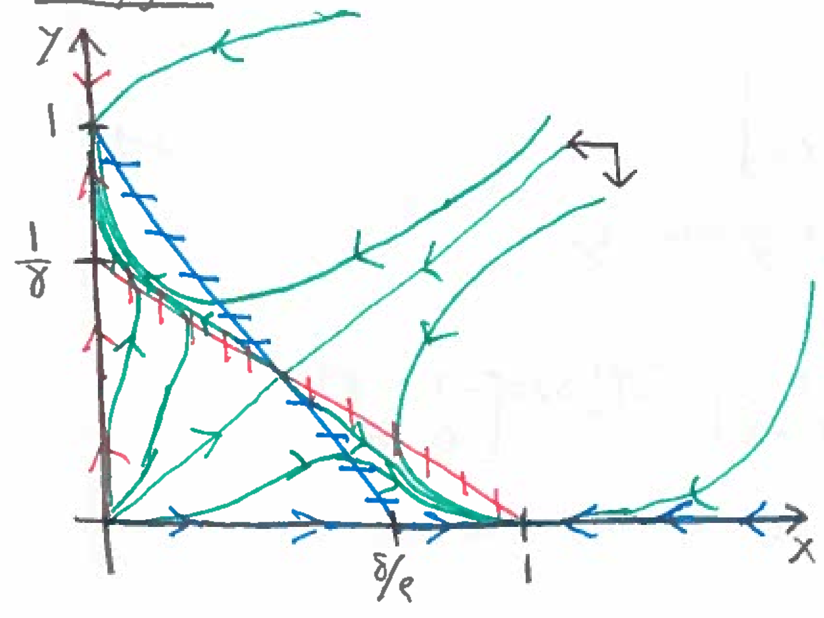
$$\Rightarrow x = \frac{\delta(1-\gamma)}{\delta - \rho\gamma}$$

The other fixed point is given by

$$\left(\frac{\delta(1-\gamma)}{\delta-\rho\gamma}, -\frac{\rho}{\delta}\frac{\delta(1-\gamma)}{\delta-\rho\gamma}+1\right) = \left(\frac{\delta(1-\gamma)}{\delta-\rho\gamma}, \frac{\delta-\rho}{\delta-\rho\gamma}\right)$$

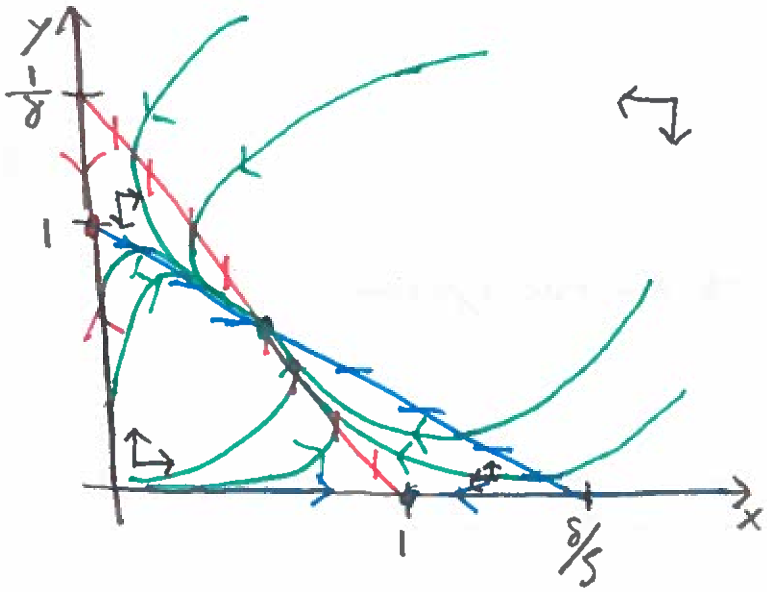
The phase portraits are thus given by:

$\underline{\gamma > 1,\ \rho > \delta}$



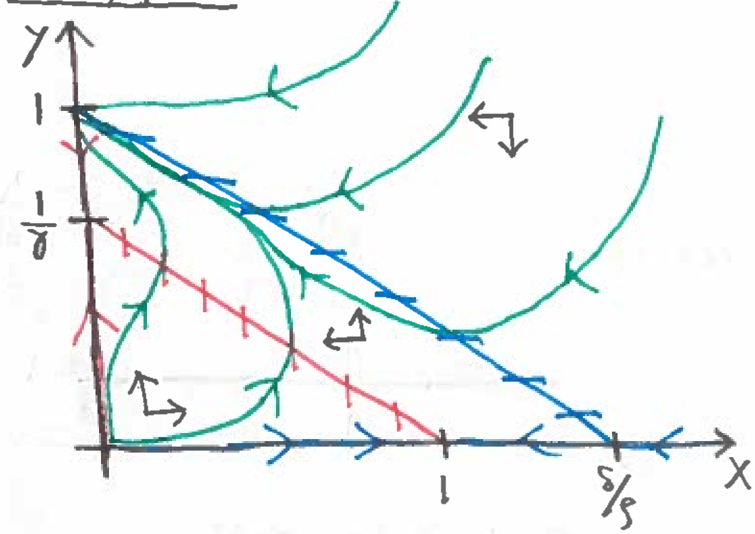
In this case one population goes extinct while the other reaches carraying capacity. Which species goes extinct depends on initial conditions.

$\gamma < 1,\ \rho < \delta$



In this case both species coexist.

$\gamma > 1,\ \rho < \delta$



In this case one species goes extinct while the other goes to carraying capacity.

#4.

Analyze the following model of a fish hatchery:

$$\dot{E} = a(pqEN - cE)$$
$$\dot{N} = rN\left(1 - \frac{N}{K}\right) - qEN$$

Solution:

Expanding it follows that

$$\dot{E} = -acE + apqEN$$
$$\dot{N} = rN\left(1 - \frac{N}{K}\right) - qEN.$$

Let $y = N/K$, $\tau = rt$. Then,

$$\frac{dE}{d\tau} = -\frac{ac}{r}E + \frac{apqK}{r}Ey.$$

$$\frac{dy}{d\tau} = y(1-y) - \frac{q}{r}Ey$$

Let $\alpha = ac/r$, $\beta = \frac{apqK}{r}$, $\gamma = \frac{q}{r}$. We obtain the system

$$\frac{dE}{d\tau} = -\alpha E + \beta E y$$

$$\frac{dy}{d\tau} = y(1-y) - \gamma E y$$

The nullclines are given by

<u>N1</u>: $E = 0$ $(\dot{E} = 0)$

<u>N2</u>: $y = \alpha/\beta$ $(\dot{E} = 0)$

<u>N3</u>: $y = 0$ $(\dot{y} = 0)$

<u>N4</u>: $y = 1 - \gamma E$ $(\dot{y} = 0)$.

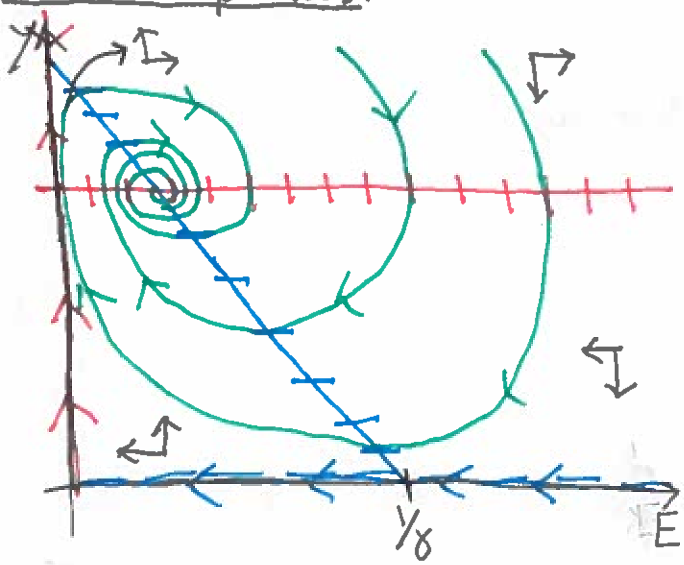
It follows that N4 and N2 intersect if

$$\frac{\alpha}{\beta} = 1 - \gamma E$$

$$\Rightarrow E = \frac{1}{\gamma} - \frac{\alpha}{\beta} = \frac{\beta - \gamma\alpha}{\gamma\beta}.$$

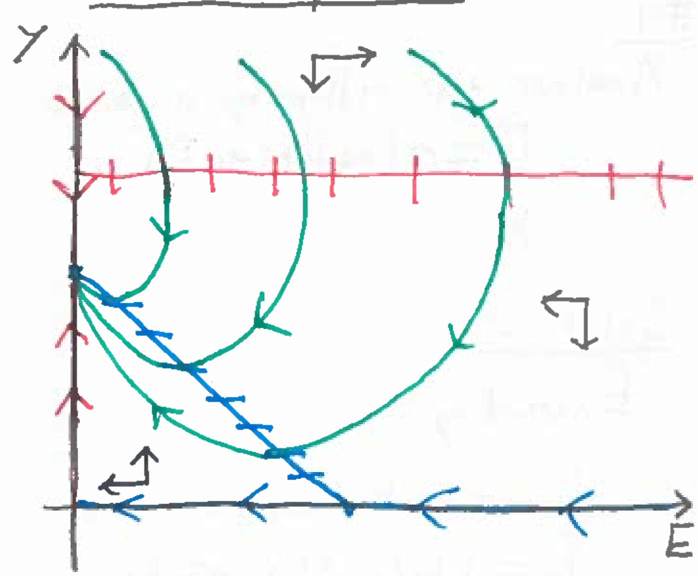
This generates two cases:

<u>Case 1 ($\beta > \gamma\alpha$)</u>:



In this case the fish pop. and effort stabalize.

<u>Case 2 ($\beta < \gamma\alpha$)</u>:



In this case effort goes to zero as it is too difficult to catch fish.

#5.

Analyze the following system:

$$\dot{x} = xy$$
$$\dot{y} = x^2 - y$$

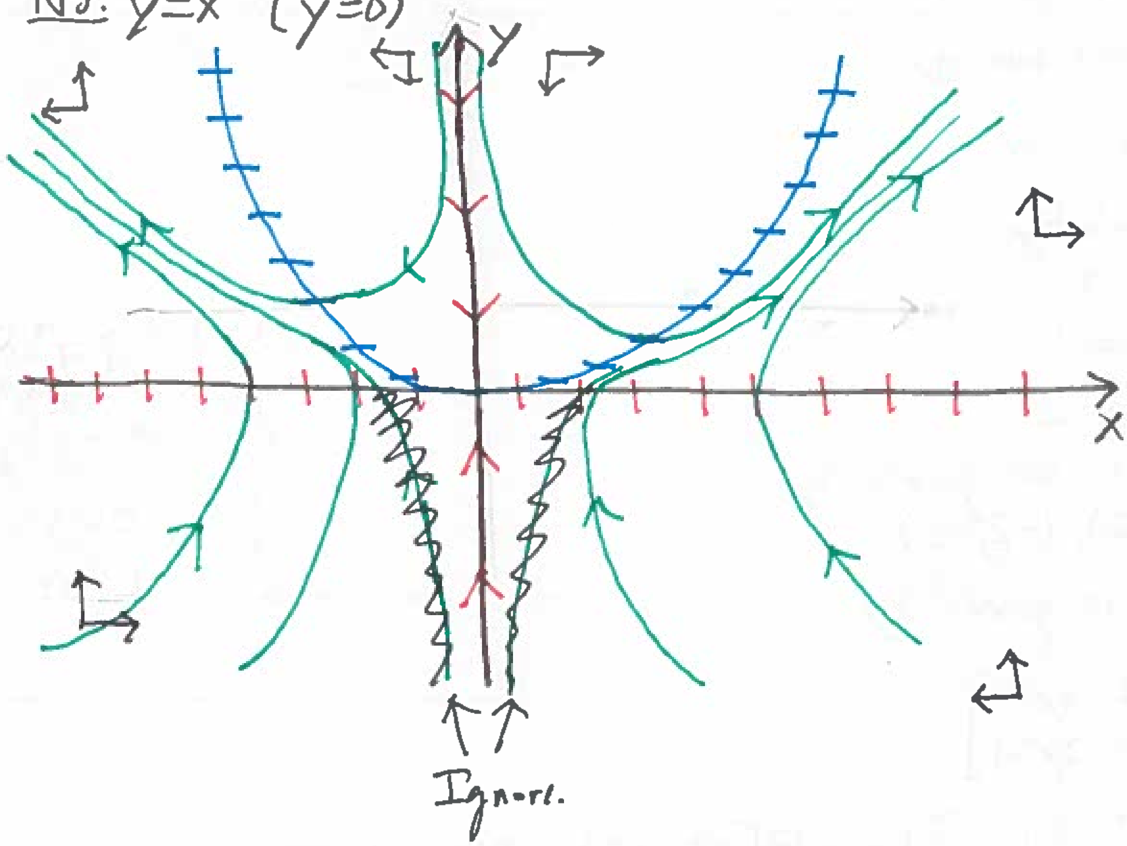
Solution:

The nullclines are

N1: $y = 0$ $(\dot{x} = 0)$

N2: $x = 0$ $(\dot{x} = 0)$

N3: $y = x^2$ $(\dot{y} = 0)$



Graduate Problem

Analyze the system

$$\dot{x} = y^3 - 4x$$
$$\dot{y} = y^3 - y - 3x.$$

Solution:

The nullclines are given by:

N1: $x = \frac{1}{4}y^3$ $(\dot{x}=0)$

N2: $x = \frac{1}{3}(y^3 - y)$ $(\dot{y}=0)$

The fixed points satisfy

$$\frac{1}{4}y^3 = \frac{1}{3}(y^3 - y)$$
$$\Rightarrow 3y^3 = 4y^3 - 4y$$
$$\Rightarrow y^3 - 4y = 0$$
$$\Rightarrow y(y^2 - 4) = 0$$
$$\Rightarrow y = 0, \; y = \pm 2$$

The fixed points are given by

$$(0,0), (2,2), (-2,-2).$$

The Jacobian is given by:

$$J(x,y) = \begin{bmatrix} -4 & 3y^2 \\ -3 & 3y^2 - 1 \end{bmatrix}$$

$$\Rightarrow J(0,0) = \begin{bmatrix} -4 & 0 \\ -3 & -1 \end{bmatrix}, \; J(2,2) = \begin{bmatrix} -4 & 12 \\ -3 & 11 \end{bmatrix}, \; J(-2,-2) = \begin{bmatrix} -4 & 12 \\ -3 & 11 \end{bmatrix}$$

Consequently, $(0,0)$ is stable while $(-2,-2)$ and $(2,2)$ are saddles.
Moreover, if we define $z = y - x$ it follows that

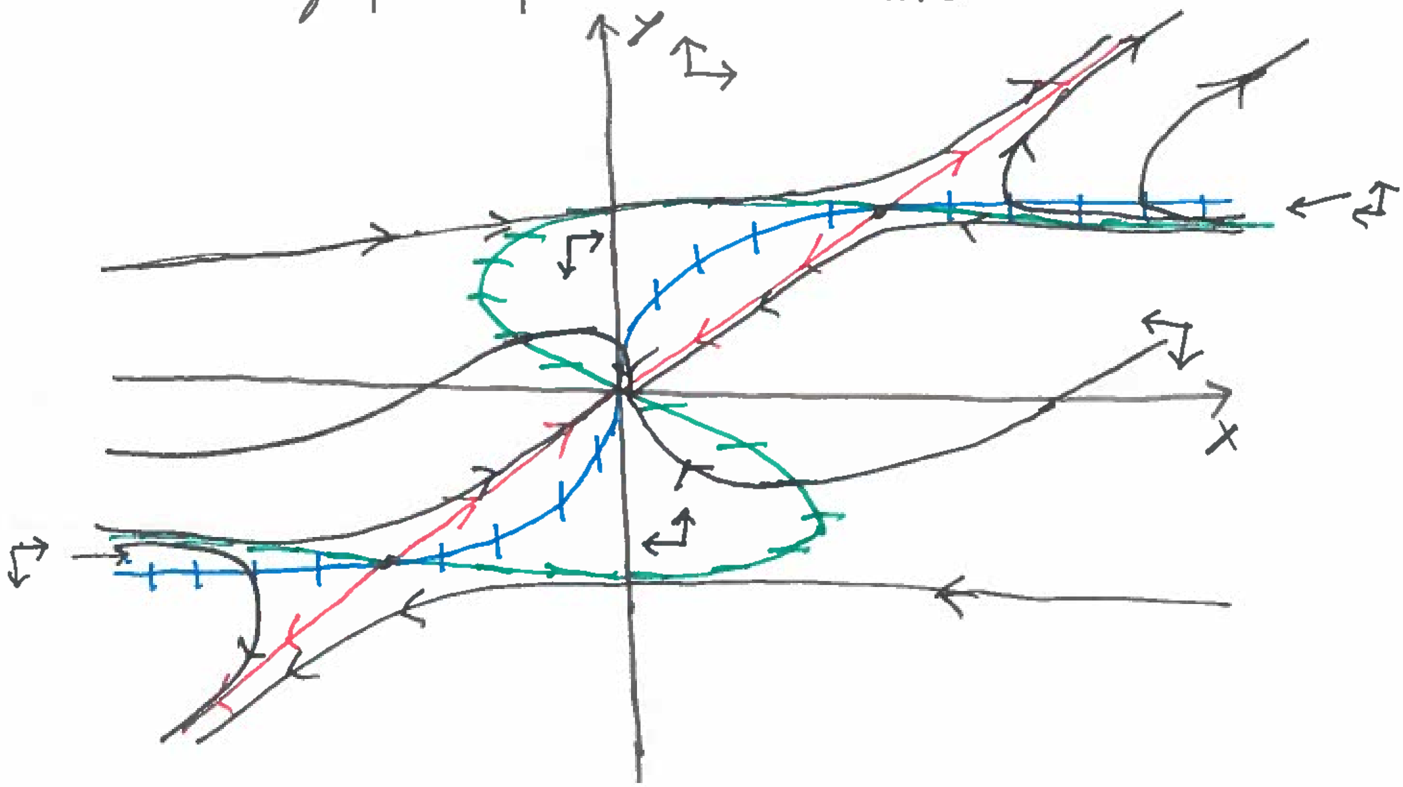
$$\dot{z} = \dot{y} - \dot{x} = -y + x = -z.$$

Consequently,

$$\lim_{t \to \infty} z = 0$$

Which implies $|y(t) - x(t)| \longrightarrow 0$.

The resulting phase portrait is therefore.



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