Homework 6

Mathematical Modeling

Due: October 24, 2018

1 Problems for Everybody

1. Consider the following system of differential equations in Cartesian coordinates

$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases}.$$

Show that if the system is converted to polar coordinates (r,θ) that

$$\begin{cases} \dot{r} = \cos(\theta)\dot{x} + \sin(\theta)\dot{y} \\ \dot{\theta} = r^{-1}\left(\cos(\theta)\dot{y} - \sin(\theta)\dot{x}\right) \end{cases}$$

Hint: Start with the relationships $r^2 = x^2 + y^2$ and $\tan(\theta) = y/x$ and differentiate with respect to time.

2. Consider the following system of differential equations in polar coordinates

$$\begin{cases} \dot{r} = f(r,\theta) \\ \dot{\theta} = g(r,\theta) \end{cases}.$$

Show that if the system is converted to Cartesian coordinates (x,y) that

$$\begin{cases} \dot{x} = \dfrac{x}{\sqrt{x^2+y^2}}\dot{r} - y\dot{\theta} \\ \dot{y} = \dfrac{y}{\sqrt{x^2+y^2}}\dot{r} + x\dot{\theta} \end{cases}.$$

3. Consider the following system in polar coordinates

$$\begin{cases} \dot{r} = -r \\ \dot{\theta} = \dfrac{1}{\ln(r)} \end{cases}.$$

   (a) Show that $r(t) \to 0$ as $t \to \infty$ and $|\theta(t)| \to \infty$ as $t \to \infty$.

   (b) Convert this system to Cartesian coordinates.

   (c) Show that the linearized system predicts that the origin is a stable star. However, based off of part (a) how would you classify the origin?

4. Consider the system $\dot{x} = -y - x^3$ and $\dot{y} = x$. Show that the origin is a spiral, although the linearization predicts a center.

5. Consider the system $\ddot{x} = x - x^2$.
   (a) Find and classify the fixed points.
   (b) Sketch the phase portrait.
   (c) Find an equation for the homoclinic orbit that separates closed and nonclosed trajectories.
6. Sketch the phase portrait for the system $\ddot{x} = ax - x^2$ for $a < 0$, $a = 0$, and $a > 0$.
7. Plot the phase portraits of the following gradient systems $\dot{\mathbf{x}} = -\nabla V(\mathbf{x})$.
   (a) $V = x^2 + y^2$
   (b) $V = x^2 - y^2$
   (c) $V = e^x \sin(y)$
8. Show that the system $\dot{x} = y - x^3$, $\dot{y} = -x - y^3$ has no closed orbits, by constructing a Liapunov function $V = ax^2 + by^2$ with suitable a, b.
9. Consider the system $\dot{x} = x^2 - y - 1$, $\dot{y} = y(x - 2)$.
   (a) Show that there are three fixed points and classify them.
   (b) By considering three straight lines through pairs of fixed points, show that there are no closed orbits.
   (c) Sketch the phase portrait.

2 Problems for MST 651 students only. Students in MST 351 can complete these problems for extra credit

1. Consider a glider flying at speed v at an angle θ to the horizontal. Its motion is governed by the dimensionless equations:

$$\begin{cases} \dot{v} = -\sin(\theta) - Dv^2 \\ v\dot{\theta} = -\cos(\theta) + v^2 \end{cases},$$

   where the trigonometric terms represent the effects of gravity and the v^2 terms represent the effect of drag.
   (a) Suppose there is no drag ($D = 0$). Show that $v^3 - 3v\cos(\theta)$ is a conserved quantity. Sketch the phase portrait in this case and interpret your results. What does the flight path of the glider look like?
   (b) Investigate the case of positive drag ($D > 0$).