Homework 6

Mathematical Modeling

Due: October 24, 2018

1 Problems for Everybody

1. Consider the following system of differential equations in Cartesian coordinates

$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases}.$$

Show that if the system is converted to polar coordinates (r,θ) that

$$\begin{cases} \dot{r} = \cos(\theta)\dot{x} + \sin(\theta)\dot{y} \\ \dot{\theta} = r^{-1}\left(\cos(\theta)\dot{y} - \sin(\theta)\dot{x}\right) \end{cases}$$

Hint: Start with the relationships $r^2 = x^2 + y^2$ and $\tan(\theta) = y/x$ and differentiate with respect to time.

✓ 2. Consider the following system of differential equations in polar coordinates

$$\begin{cases} \dot{r} = f(r,\theta) \\ \dot{\theta} = g(r,\theta) \end{cases}.$$

Show that if the system is converted to Cartesian coordinates (x,y) that

$$\begin{cases} \dot{x} = \dfrac{x}{\sqrt{x^2+y^2}}\dot{r} - y\dot{\theta} \\ \dot{y} = \dfrac{y}{\sqrt{x^2+y^2}}\dot{r} + x\dot{\theta} \end{cases}.$$

3. Consider the following system in polar coordinates

$$\begin{cases} \dot{r} = -r \\ \dot{\theta} = \dfrac{1}{\ln(r)} \end{cases}.$$

(a) Show that $r(t) \to 0$ as $t \to \infty$ and $|\theta(t)| \to \infty$ as $t \to \infty$.

(b) Convert this system to Cartesian coordinates.

(c) Show that the linearized system predicts that the origin is a stable star. However, based off of part (a) how would you classify the origin?

✓ 4. Consider the system $\dot{x} = -y - x^3$ and $\dot{y} = x$. Show that the origin is a spiral, although the linearization predicts a center.

5. Consider the system $\ddot{x} = x - x^2$.
   (a) Find and classify the fixed points.
   (b) Sketch the phase portrait.
   (c) Find an equation for the homoclinic orbit that separates closed and nonclosed trajectories.

6. Sketch the phase portrait for the system $\ddot{x} = ax - x^2$ for $a < 0$, $a = 0$, and $a > 0$.

7. Plot the phase portraits of the following gradient systems $\dot{\mathbf{x}} = -\nabla V(\mathbf{x})$.
   (a) $V = x^2 + y^2$
   (b) $V = x^2 - y^2$
   (c) $V = e^x \sin(y)$

8. Show that the system $\dot{x} = y - x^3$, $\dot{y} = -x - y^3$ has no closed orbits, by constructing a Liapunov function $V = ax^2 + by^2$ with suitable a, b.

9. Consider the system $\dot{x} = x^2 - y - 1$, $\dot{y} = y(x - 2)$.
   (a) Show that there are three fixed points and classify them.
   (b) By considering three straight lines through pairs of fixed points, show that there are no closed orbits.
   (c) Sketch the phase portrait.

2 Problems for MST 651 students only. Students in MST 351 can complete these problems for extra credit

1. Consider a glider flying at speed v at an angle θ to the horizontal. Its motion is governed by the dimensionless equations:

$$\begin{cases} \dot{v} = -\sin(\theta) - Dv^2 \\ v\dot{\theta} = -\cos(\theta) + v^2 \end{cases}$$

   where the trigonometric terms represent the effects of gravity and the v^2 terms represent the effect of drag.

   (a) Suppose there is no drag ($D = 0$). Show that $v^3 - 3v\cos(\theta)$ is a conserved quantity. Sketch the phase portrait in this case and interpret your results. What does the flight path of the glider look like?
   (b) Investigate the case of positive drag ($D > 0$).

Homework #6

#2.

Consider the following system of differential equations in polar coordinates

$$\begin{cases} \dot{r} = f(r,\theta) \\ \dot{\theta} = g(r,\theta) \end{cases}.$$

Show that if the system is converted to Cartesian coordinates (x,y) that

$$\begin{cases} \dot{x} = \dfrac{x}{\sqrt{x^2+y^2}}\dot{r} - y\dot{\theta} \\ \dot{y} = \dfrac{y}{\sqrt{x^2+y^2}}\dot{r} + x\dot{\theta} \end{cases}$$

Solution:

Since $x = r\cos\theta$ and $y = r\sin\theta$ it follows that

$$\begin{cases} \dot{x} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta} \\ \dot{y} = \dot{r}\sin\theta + r\cos\theta\,\dot{\theta} \end{cases}$$

$$\Rightarrow \begin{cases} \dot{x} = \dfrac{r\cos\theta}{r}\dot{r} - r\sin\theta\,\dot{\theta} \\ \dot{y} = \dfrac{r\sin\theta}{r}\dot{r} + r\cos\theta\,\dot{\theta} \end{cases}$$

$$\Rightarrow \begin{cases} \dot{x} = \dfrac{x}{\sqrt{x^2+y^2}}\dot{r} - y\dot{\theta} \\ \dot{y} = \dfrac{y}{\sqrt{x^2+y^2}}\dot{r} + x\dot{\theta} \end{cases}$$

#4.

Consider the system $\dot{x} = -y - x^3$ and $\dot{y} = x$. Show that the origin is a spiral, although the linearization predicts a center.

Solution:

The Jacobian at $(0,0)$ is $J(0,0) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ which predicts a Center. However, converting to polar coordinates it follows that

$$\begin{cases} \dot{r} = \cos\theta\,\dot{x} + \sin\theta\,\dot{y} \\ \dot{\theta} = r^{-1}(\cos\theta\,\dot{y} - \sin\theta\,\dot{x}) \end{cases}$$

$$\Rightarrow \begin{cases} \dot{r} = -r\cos\theta\sin\theta - r^3\cos^4\theta + r\cos\theta\sin\theta \\ \dot{\theta} = r^{-1}(r\cos^2\theta + r\sin^2\theta + r^3\cos^3\theta\sin\theta) \end{cases}$$

$$\Rightarrow \begin{cases} \dot{r} = -r^3\cos^4\theta \\ \dot{\theta} = 1 + r^2\cos^3\theta\sin\theta \end{cases}$$

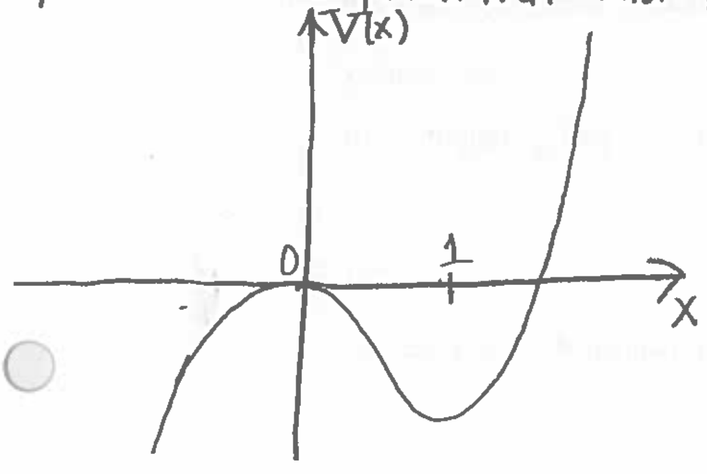
Therefore, if $r < 1$ it follows that $\dot{r} < 0$ and $\dot{\theta} > 0$ which corresponds to a stable spiral.

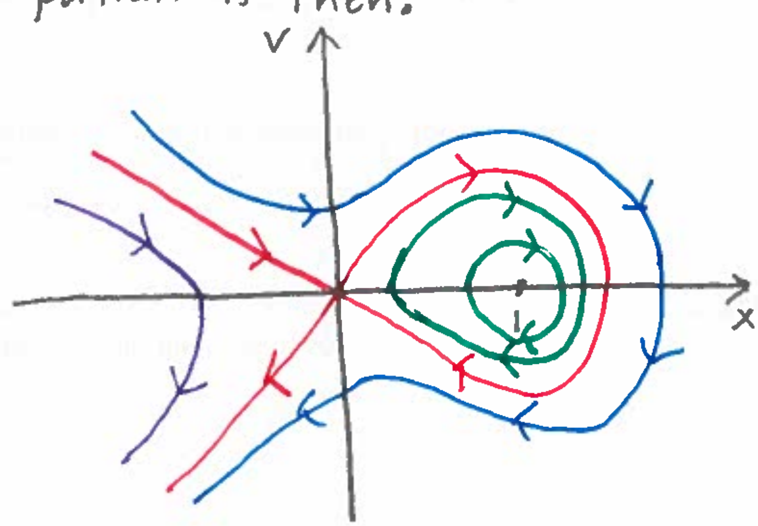
#5.

Consider the system $\ddot{x} = x - x^2$. Sketch the phase portrait and find an equation for the homoclinic orbit.

Solution:

This a conservative system with potential $V = \frac{x^3}{3} - \frac{x^2}{2}$. The resulting plot of the potential and phase portrait is then:





Since $V(0)=0$ it follows that $E(0,0)=0$ where $E(x,v)$ is the energy defined by $E(x,v)=\frac{1}{2}v^2+\frac{x^3}{3}-\frac{x^2}{2}$. Consequently, since the homoclinic orbit passes through $(0,0)$ it follows that

$$0=\frac{1}{2}v^2+\frac{x^3}{3}-\frac{x^2}{2}$$

$$\Longrightarrow v=\pm\sqrt{x^2-\frac{2}{3}x^3}.$$

#8.

Show that the system $\dot{x}=y-x^3$, $\dot{y}=-x-y^3$ has no closed orbits, by constructing Liaponov function $V=ax^2+by^2$ with suitable a,b.

Solution:

Calculating it follows that

$$\begin{aligned}\dot{V}&=2ax\dot{x}+2by\dot{y}\\&=2ax(y-x^3)+2by(-x-y^3)\\&=2ax-2ax^4-2byx-2by^4\end{aligned}$$

If $a=b>0$ it follows that

$$\dot{V}=-2(ax^4+by^4)<0.$$

#9

Consider the system $\dot{x}=x^2-y-1$, $\dot{y}=y(x-2)$.

a.) Show that there are three fixed points and classify them.

Solution:

The nullclines for this system are given by

N1: $y=x^2-1$, $\dot{x}=0$

N2: $y=0$, $\dot{y}=0$

N3: $x=2$, $\dot{y}=0$

Consequently, the fixed points are $(-1,0)$, $(1,0)$, and $(2,3)$. The Jacobian is thus given by:

$$J=\begin{bmatrix} 2x & -1 \\ y & x-2 \end{bmatrix}$$

$$\Rightarrow J(-1,0)=\begin{bmatrix} -2 & -1 \\ 0 & -3 \end{bmatrix},\ J(1,0)=\begin{bmatrix} 2 & -1 \\ 0 & -1 \end{bmatrix},\ J(2,3)=\begin{bmatrix} 4 & -1 \\ 3 & 0 \end{bmatrix}$$

$\Rightarrow$ $(-1,0)$ is a stable node
$(1,0)$ is a saddle
$(2,3)$ is an unstable node.

b.) By considering three straight lines through pairs of fixed points, show that there are no closed orbits.

Solution:

The lines connecting the fixed points are given by:

1. $y=0$
2. $y=x+1$
3. $y=3x-3$

Furthermore,

$$\frac{dy}{dx}=\frac{\dot{y}}{\dot{x}}=\frac{y(x-2)}{x^2-y-1}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{y=0}=0$$

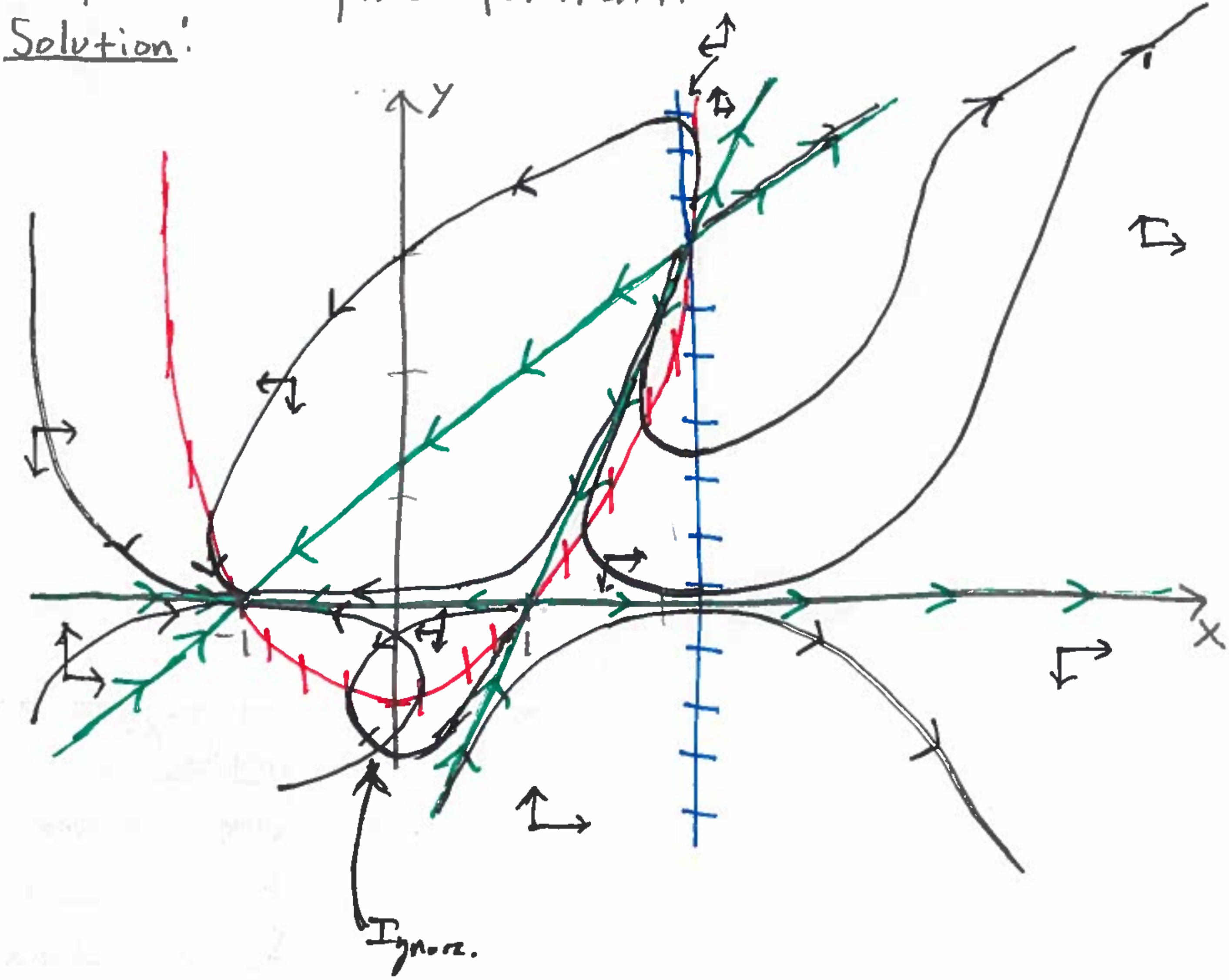
$$\left.\frac{dy}{dx}\right|_{y=x+1}=\frac{(x+1)(x-2)}{(x^2-x-1-1)}=\frac{(x+1)(x-2)}{(x+1)(x-2)}=1$$

$$\left.\frac{dy}{dx}\right|_{y=3x-3}=\frac{(3x-3)(x-2)}{(x^2-3x+2)}=\frac{3(x-1)(x-2)}{(x-1)(x-2)}=3.$$

Therefore, the lines connecting the fixed points are invariant submanifolds. Consequently, no closed orbits can encircle the fixed points else it would violate existence and uniqueness.

c) Sketch the phase portrait.

Solution:



Graduate Problem:

#1

Consider a glider flying at speed v at an angle θ to the horizontal. Its motion is governed by the dimensionless equations:

$$\begin{cases} \dot{v} = -\sin\theta - Dv^2 \\ v\dot{\theta} = -\cos\theta + v^2 \end{cases}$$

a.) Suppose there is no drag ($D=0$), show that $v^3 - 3v\cos\theta$ is a conserved quantity. Sketch the phase portrait in this case and interpret your results.

Solution:

Let $E = v^3 - 3v\cos\theta$. Therefore,

$$\dot{E} = 3v^2\dot{v} - 3\dot{v}\cos\theta + 3v\sin\theta \cdot \dot{\theta}$$

$$= 3v^2(-\sin\theta) + 3\cos\theta\sin\theta + \frac{3v\sin\theta\,(-\cos\theta + v^2)}{v}$$

$$= 0.$$

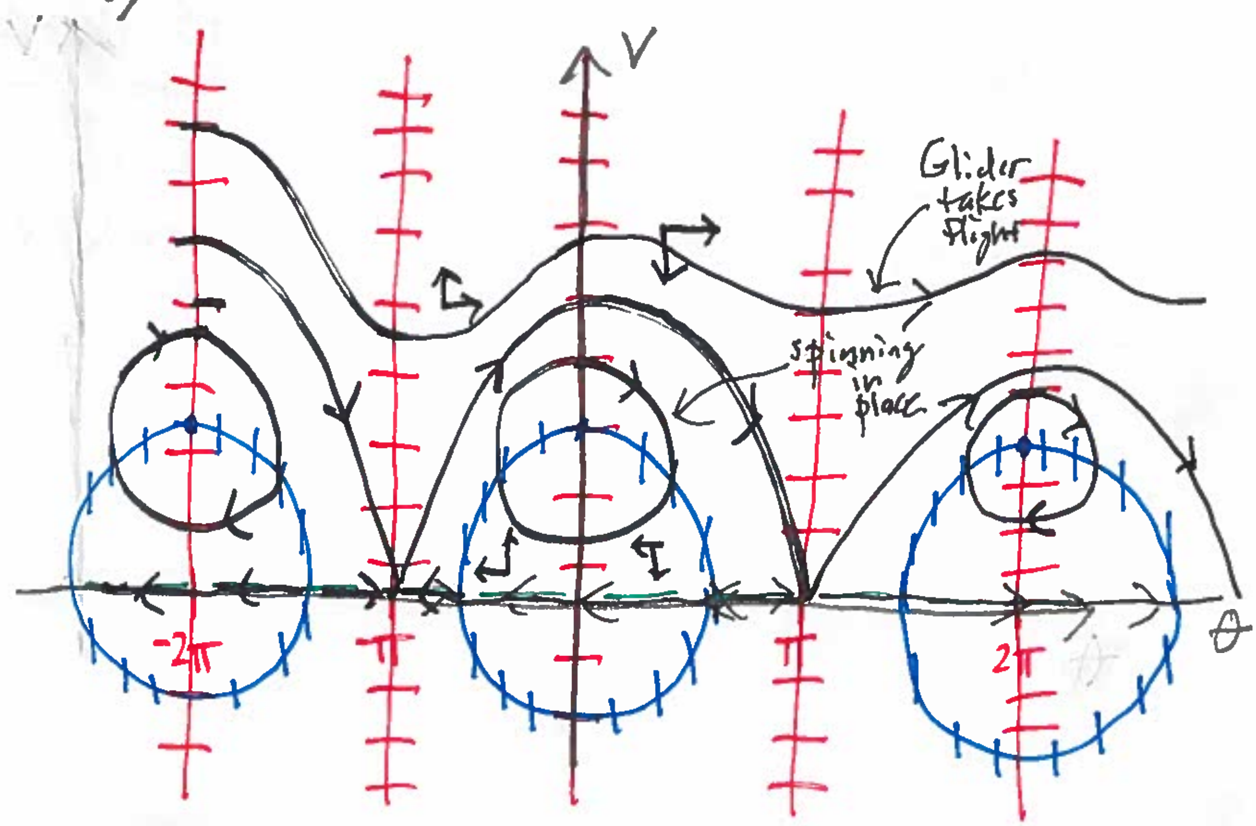
Consequently, the solution curves are contours of E.

For the phase portrait the nullclines are given by:

N1: $\theta = n\pi$ $\quad(\dot{v} = 0)$

N2: $v = \pm\sqrt{\cos\theta}$, $\quad(\dot{\theta} = 0)$.

Since the system is conservative the fixed points can only be saddles and centers. The phase portrait is thus given by.



In this case the glider's motion essentially spins in place (closed orbits) or takes flight (unbounded orbits).

b.) Investigate the case of positive drag ($D>0$).

Solution:

Lets first look at the previously conserved quantity $E=v^3-3v\cos\theta$. Differentiating it follows that

$$\dot{E}=3v^2\dot{v}-3\dot{v}\cos\theta+3v\sin\theta\,\dot{\theta}$$
$$=3v^2(-\sin\theta-Dv^2)-3(-\sin\theta-Dv^2)\cos\theta+3v\sin\theta\cdot\left(\frac{-\cos\theta}{v}+v\right)$$
$$=-3Dv^4+3Dv^2\cos\theta+3v^2\sin\theta.$$

This yields very little of use. We instead look at the nullclines and fixed points.

N1: $v=\pm\sqrt{\frac{-\sin\theta}{D}}$ ($\dot{v}=0$)

N2: $v=\pm\sqrt{\cos\theta}$ ($\dot{\theta}=0$)

Also, $v=0$ is in some sense a nullcline along which $\dot{\theta}=\pm\infty$. The fixed points occur when

$$\frac{-\sin\theta}{D}=\cos\theta$$

$$\Rightarrow \tan\theta=-D. \Rightarrow \theta^*=-\tan^{-1}(D) \text{ and } v^*=\left(\frac{1}{\sqrt{1+D^2}}\right)^{1/2}$$

The Jacobian matrix is given by

$$J=\begin{bmatrix} -\frac{\sin\theta}{v} & \frac{\cos\theta}{v^2}+1 \\ -\cos\theta & -2Dv \end{bmatrix}$$

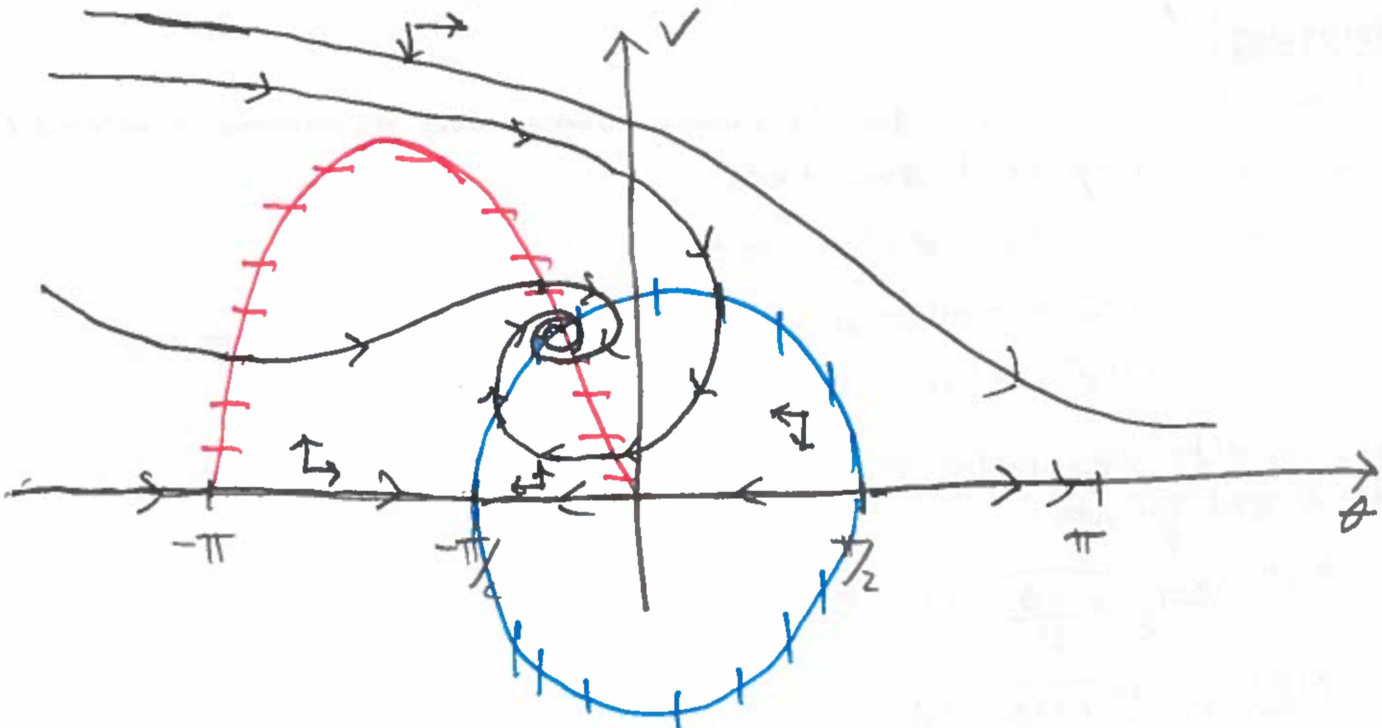
$$J\left(-\tan^{-1}(D), \frac{1}{\sqrt{1+D^2}}\right)=\begin{bmatrix} \frac{D}{(1+D^2)^{1/4}} & 2 \\ \frac{-1}{\sqrt{1+D^2}} & \frac{-2D}{(1+D^2)^{1/4}} \end{bmatrix}$$

Now, $Det(J)=\frac{2(1+d^2)}{\sqrt{1+d^2}}$ and $Tr(J)=\frac{-3D}{(1+D^2)^{1/4}}$. Consequently,

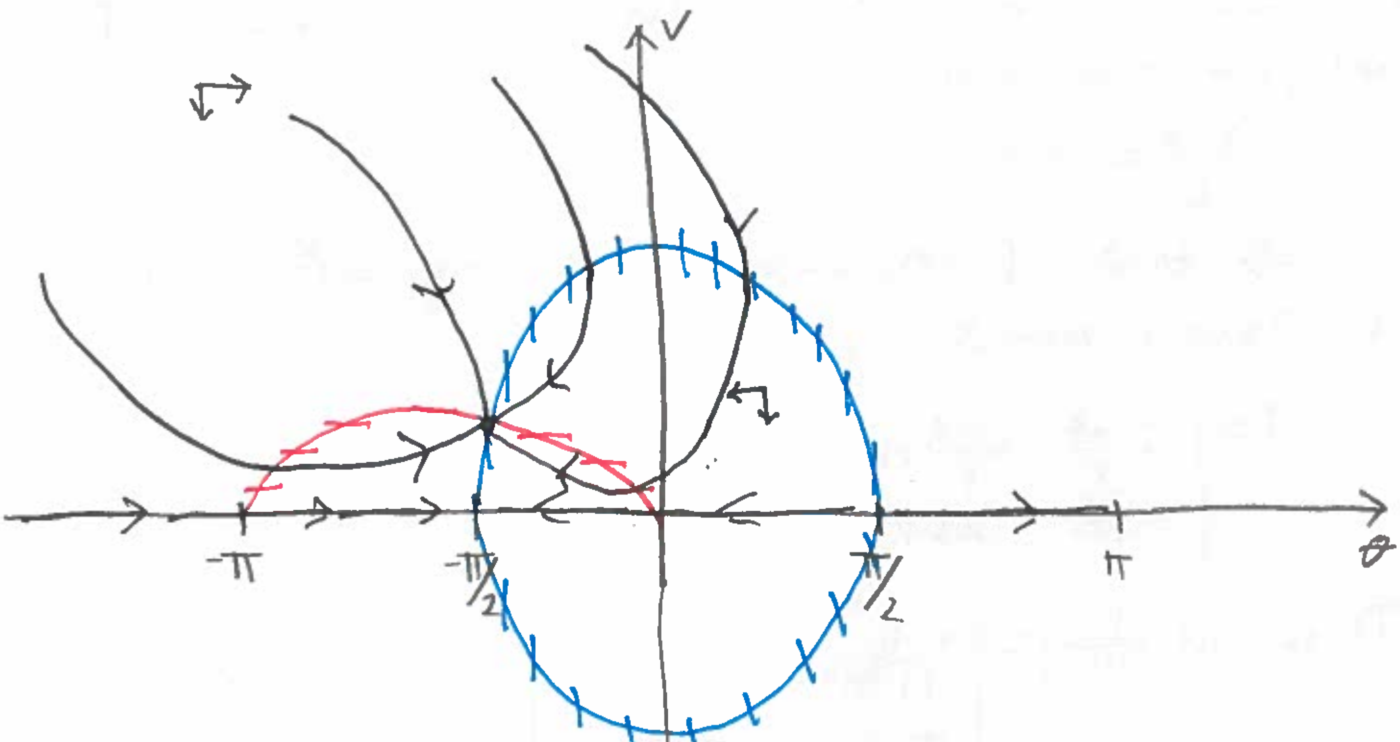
$$Tr(J)^2-4Det(J)=\frac{-8+D^2}{\sqrt{1+D^2}}$$

Consequently, if $D<2\sqrt{2}$ the fixed point is a stable spiral but if $D>2\sqrt{2}$ it is a stable node. This yields two distinct phase portraits.

$\underline{D < 2\sqrt{2}}$



$\underline{D > 2\sqrt{2}}$



In either case the glider becomes locked in a negative angle, i.e. it starts diving.