Homework 7

Mathematical Modeling

Due: October 31, 2018

1 Problems for Everybody

1. Consider the following system of differential equations:

$$\begin{cases} \dot{x} = x - y - x(x^2 + 5y^2) \\ \dot{y} = x + y - y(x^2 + y^2) \end{cases}.$$

   (a) Classify the fixed point at the origin.

   (b) Rewrite the system in polar coordinates.

   (c) Determine the circle of maximum radius centered at the such that all trajectories have a radially outward component on it.

   (d) Determine the circle of minimum radius centered at the origin such that all trajectories have a radially inward component on it.

   (e) Prove that the system has a limit cycle.

2. Prove that the system

$$\begin{cases} \dot{x} = x - y - x^3 \\ \dot{y} = x + y - y^3 \end{cases}$$

   has a periodic solution.

3. Consider the system

$$\begin{cases} \dot{x} = x(1 - 4x^2 - y^2) - \frac{1}{2}y(1 + x) \\ \dot{y} = y(1 - 4x^2 - y^2) + 2x(1 + x) \end{cases}.$$

   (a) Show that the origin is an unstable fixed point.

   (b) By considering $\dot{V}$, where $V = (1 - 4x^2 - y^2)^2$, show that all trajectories approach the ellipse $4x^2 + y^2 = 1$ as $t \to \infty$.

4. Show that the system

$$\begin{cases} \dot{x} = -x - y + x(x^2 + 2y^2) \\ \dot{y} = x - y + y(x^2 + 2y^2) \end{cases}$$

   has at least one periodic solution.

5. For each of the following systems, locate the fixed points and calculate the index.

   (a) $\dot{x} = x^2$, $\dot{y} = y$

   (b) $\dot{x} = y^3$, $\dot{y} = x$

   (c) $\dot{x} = y - x$, $\dot{y} = x^2$

   (d) $\dot{x} = xy$, $\dot{y} = x + y$

6. A closed orbit in the phase plane encircles S saddles, N nodes, F spirals, and C centers. Show that $N + F + C = 1 + S$.

7. A smooth vector field on the phase plane is known to have exactly three closed orbits. Two of the cycles, say C_1 and C_2, lie inside the third cycle C_3. However, C_1 does not lie inside C_2, nor vice-versa.
   (a) Sketch a possible arrangement of the three cycles.
   (b) Show that there must be at least one fixed point in the region bounded by C_1, C_2, C_3.
8. A smooth vector field on the phase plane is known to have exactly two closed trajectories, one of which lies inside the other. The inner cycle runs clockwise, and the outer one runs counterclockwise. True or False: there must be at least one fixed point in the region between the cycles. If true, prove it. If false, provide a simple counterexample.
9. Consider the following system in *polar coordinates*

$$\begin{cases} \dot{r} = r(1 + \cos(\theta)) \\ \dot{\theta} = r(1 - \cos(\theta)) \end{cases} .$$

   Sketch the phase portrait.

2 Problems for MST 651 students only. Students in MST 351 can complete these problems for extra credit

1. Consider the following system

$$\dot{\mathbf{x}} = F(\mathbf{x}),$$

   where $F : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is a smooth function. Suppose $\nabla \cdot F$ does not change sign in a simply connected region U in $\mathbb{R}^2$. Show that there are no periodic orbits inside U. **Hint:** Suppose there is one and consider the line integral of F along this curve and apply Green's theorem.
2. Show that if $f, g : \mathbb{R} \mapsto \mathbb{R}$ are smooth functions, then the equation

$$\begin{cases} \dot{x} = f(y) \\ \dot{y} = g(x) + y^3 \end{cases} ,$$

   has no periodic orbits.