Homework 8

Mathematical Modeling

Due: November 7, 2018

1 Problems for Everybody

1. Consider the following model for a drug prescription:

$$a_{n+1} = a_n - ka_n + b,$$

where a_n is the amount of a drug (in mg) in the bloodstream after administration of n dosages hourly.

   (a) Discuss the meaning of the model parameters k and b. What can you say about their size and sign?

   (b) Find the fixed points of the model and determine their stability.

   (c) Perform a cobwebbing analysis for this model. What happens to the amount of drug in the bloodstream in the long run? How does the result depend on the model parameters?

   (d) How should b be chosen to ensure that the drug is effective, but not toxic?

2. In this problem you can use Mathematica to help you with the calculations. You just need to include a copy of your code. Consider the discrete logistic equation:

$$x_{n+1} = rx_n(1 - x_n).$$

   (a) Compute $f^2(x)$, $f^3(x)$, and $f^4(x)$.

   (b) Find the fixed points of $f^2(x)$, $f^3(x)$, and $f^4(x)$. At what values of r, if any, does a 2-cycle, 3-cycle, or 4-cycle appear. For what values of r are these cycles stable? Unstable?

3. Consider the following discrete dynamical system

$$x_{n+1} = ax_n e^{-x_n} \text{ and } x_0 > 0.$$

where $a > 0$.

   (a) Find the fixed points and analyze their stability.

   (b) Show that for all n, $x_n \geq 0$.

   (c) Using Matlab, or some other software, create the orbital bifurcation diagram for this problem. Print and include a copy of this diagram with your homework. Compare and contrast this orbital bifurcation diagram with the one from the logistic equation.

#1

Consider the following model for a drug prescription:

$$a_{n+1} = a_n - Ka_n + b$$

where a_n is the amount of drug in the bloodstream after n dosages.

a.) Discuss the meaning of the model parameters K and b. What can you say about their size and sign?

Solution:

K is the fraction that is removed from the bloodstream by absorption. b is the amout added to bloodstream by the dosage. Both of these constants must be positive.

b.) Find the fixed points and determine their stability.

Solution:

Fixed points satisfy:

$$a^* = a^* - Ka^* + b$$

$$\Rightarrow a^* = b/K.$$

If we let $f(a) = a - Ka + b$ then $f'(a) = 1 - K$. Consequently, a^* is stable if $0 \leq K \leq 2$.

d.) How should b be chosen to ensure that the drug is effective but not toxic?

Solution:

Let $T > 0$ denote a toxicity level. Then, we need $b/K < T$ and thus $b < TK$.

Homework #8, Problem #2

Consider the discrete logistic equation: $x_{n+1} = r\, x_n(1 - x_n)$

Part (a).

Compute $f^2(x)$, $f^3(x)$, $f^4(x)$.

In[162]:=
```
f[x_] := r * x * (1 - x);
g[x_] := f[f[x]];
h[x_] := f[f[f[x]]];
k[x_] := f[f[f[f[x]]]];
```

The computations are as follows:

In[167]:=
```
FullSimplify[g[x]]
FullSimplify[h[x]]
FullSimplify[k[x]]
```

Out[167]= $-r^2\,(-1+x)\,x\,(1+r\,(-1+x)\,x)$

Out[168]= $r^3\,(1-x)\,x\,(1+r\,(-1+x)\,x)\,\left(1+r^2\,(-1+x)\,x\,(1+r\,(-1+x)\,x)\right)$

Out[169]= $r^4\,(1-x)\,x\,(1+r\,(-1+x)\,x)\,\left(1+r^2\,(-1+x)\,x\,(1+r\,(-1+x)\,x)\right)\left(1+r^3\,(-1+x)\,x\,(1+r\,(-1+x)\,x)\,\left(1+r^2\,(-1+x)\,x\,(1+r\,(-1+x)\,x)\right)\right)$

Part (b).

Find the fixed points of $f^2(x)$, $f^3(x)$, $f^4(x)$. At what values of r, if any, does a 2-cycle, 3-cycle, or 4-cycle appear. For what values of r are threse cycles stable? Unstable?

Two Period Orbits

In[170]:=
```
Solve[g[x] == x, x]
```

Out[170]= $\left\{\{x\to 0\},\ \left\{x\to \frac{-1+r}{r}\right\},\ \left\{x\to \frac{r+r^2-r\sqrt{-3-2r+r^2}}{2r^2}\right\},\ \left\{x\to \frac{r+r^2+r\sqrt{-3-2r+r^2}}{2r^2}\right\}\right\}$

In[172]:=
```
FullSimplify[D[g[x], x]]
```

Out[172]= $-r^2\,(-1+2x)\,(1+2r\,(-1+x)\,x)$

In[173]:=
```
gp[x_] := -r^2 (-1 + 2 x) (1 + 2 r (-1 + x) x);
```

In[179]:= $\text{Solve}\left[\text{FullSimplify}\left[\text{Abs}\left[\text{gp}\left[\frac{r+r^2-r\sqrt{-3-2r+r^2}}{2r^2}\right]\right]\right] == 1,\ r\right]$

Out[179]= $\left\{\{r\to -1\},\ \{r\to 3\},\ \left\{r\to 1-\sqrt{6}\right\},\ \left\{r\to 1+\sqrt{6}\right\}\right\}$

This tells us that the period two orbits are stable between r=3 and $1+\sqrt{6}$.

Three Period Orbits

In[180]:=
```
Solve[h[x] == x, x]
```

Out[180]=

$$\left\{\{x \to 0\}, \left\{x \to \frac{-1+r}{r}\right\},\right.$$

$$\{x \to \text{Root}[1 + r + r^2 + (-r - 2 r^2 - 2 r^3 - r^4)\, \#1 + (r^2 + 3 r^3 + 3 r^4 + 2 r^5)\, \#1^2 + (-r^3 - 3 r^4 - 5 r^5 - r^6)\, \#1^3 + (r^4 + 4 r^5 + 3 r^6)\, \#1^4 + (-r^5 - 3 r^6)\, \#1^5 + r^6\, \#1^6\, \&,\, 1]\},$$

$$\{x \to \text{Root}[1 + r + r^2 + (-r - 2 r^2 - 2 r^3 - r^4)\, \#1 + (r^2 + 3 r^3 + 3 r^4 + 2 r^5)\, \#1^2 + (-r^3 - 3 r^4 - 5 r^5 - r^6)\, \#1^3 + (r^4 + 4 r^5 + 3 r^6)\, \#1^4 + (-r^5 - 3 r^6)\, \#1^5 + r^6\, \#1^6\, \&,\, 2]\},$$

$$\{x \to \text{Root}[1 + r + r^2 + (-r - 2 r^2 - 2 r^3 - r^4)\, \#1 + (r^2 + 3 r^3 + 3 r^4 + 2 r^5)\, \#1^2 + (-r^3 - 3 r^4 - 5 r^5 - r^6)\, \#1^3 + (r^4 + 4 r^5 + 3 r^6)\, \#1^4 + (-r^5 - 3 r^6)\, \#1^5 + r^6\, \#1^6\, \&,\, 3]\},$$

$$\{x \to \text{Root}[1 + r + r^2 + (-r - 2 r^2 - 2 r^3 - r^4)\, \#1 + (r^2 + 3 r^3 + 3 r^4 + 2 r^5)\, \#1^2 + (-r^3 - 3 r^4 - 5 r^5 - r^6)\, \#1^3 + (r^4 + 4 r^5 + 3 r^6)\, \#1^4 + (-r^5 - 3 r^6)\, \#1^5 + r^6\, \#1^6\, \&,\, 4]\},$$

$$\{x \to \text{Root}[1 + r + r^2 + (-r - 2 r^2 - 2 r^3 - r^4)\, \#1 + (r^2 + 3 r^3 + 3 r^4 + 2 r^5)\, \#1^2 + (-r^3 - 3 r^4 - 5 r^5 - r^6)\, \#1^3 + (r^4 + 4 r^5 + 3 r^6)\, \#1^4 + (-r^5 - 3 r^6)\, \#1^5 + r^6\, \#1^6\, \&,\, 5]\},$$

$$\left.\{x \to \text{Root}[1 + r + r^2 + (-r - 2 r^2 - 2 r^3 - r^4)\, \#1 + (r^2 + 3 r^3 + 3 r^4 + 2 r^5)\, \#1^2 + (-r^3 - 3 r^4 - 5 r^5 - r^6)\, \#1^3 + (r^4 + 4 r^5 + 3 r^6)\, \#1^4 + (-r^5 - 3 r^6)\, \#1^5 + r^6\, \#1^6\, \&,\, 6]\}\right\}$$

In[187]:=
```
h[x]
```

Out[187]=

$$r^3\, (1 - x)\, x\, (1 - r\, (1 - x)\, x)\, \left(1 - r^2\, (1 - x)\, x\, (1 - r\, (1 - x)\, x)\right)$$

In[188]:=
```
FullSimplify[D[h[x], x]]
```

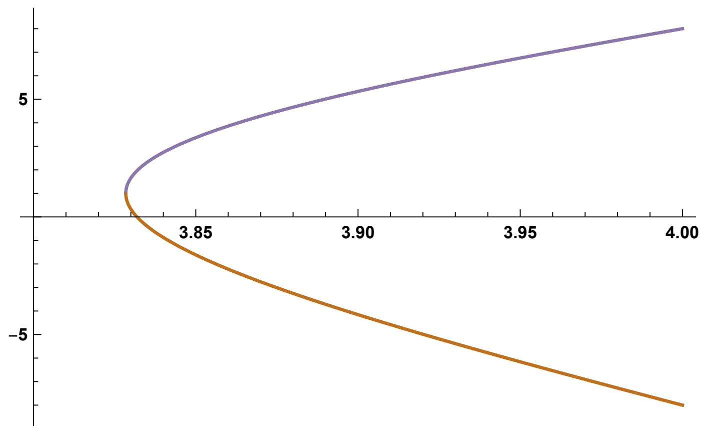
Out[188]=

$$-r^3\, (-1 + 2 x)\, (1 + 2 r\, (-1 + x)\, x\, (1 + r\, (1 + r\, (-1 + x)\, x)\, (1 + 2 r\, (-1 + x)\, x)))$$

In[189]:=
```
hp[x_] := -r^3 (-1 + 2 x) (1 + 2 r (-1 + x) x (1 + r (1 + r (-1 + x) x) (1 + 2 r (-1 + x) x)))
```

```
In[194]:= Plot[{hp[Root[1 + r + r^2 + (-r - 2 r^2 - 2 r^3 - r^4) #1 + (r^2 + 3 r^3 + 3 r^4 + 2 r^5) #1^2 +
        (-r^3 - 3 r^4 - 5 r^5 - r^6) #1^3 + (r^4 + 4 r^5 + 3 r^6) #1^4 + (-r^5 - 3 r^6) #1^5 + r^6 #1^6 &, 1]],
   hp[Root[1 + r + r^2 + (-r - 2 r^2 - 2 r^3 - r^4) #1 + (r^2 + 3 r^3 + 3 r^4 + 2 r^5) #1^2 +
        (-r^3 - 3 r^4 - 5 r^5 - r^6) #1^3 + (r^4 + 4 r^5 + 3 r^6) #1^4 + (-r^5 - 3 r^6) #1^5 + r^6 #1^6 &, 2]],
   hp[Root[1 + r + r^2 + (-r - 2 r^2 - 2 r^3 - r^4) #1 + (r^2 + 3 r^3 + 3 r^4 + 2 r^5) #1^2 +
        (-r^3 - 3 r^4 - 5 r^5 - r^6) #1^3 + (r^4 + 4 r^5 + 3 r^6) #1^4 + (-r^5 - 3 r^6) #1^5 + r^6 #1^6 &, 3]],
   hp[Root[1 + r + r^2 + (-r - 2 r^2 - 2 r^3 - r^4) #1 + (r^2 + 3 r^3 + 3 r^4 + 2 r^5) #1^2 +
        (-r^3 - 3 r^4 - 5 r^5 - r^6) #1^3 + (r^4 + 4 r^5 + 3 r^6) #1^4 + (-r^5 - 3 r^6) #1^5 + r^6 #1^6 &, 4]],
   hp[Root[1 + r + r^2 + (-r - 2 r^2 - 2 r^3 - r^4) #1 + (r^2 + 3 r^3 + 3 r^4 + 2 r^5) #1^2 +
        (-r^3 - 3 r^4 - 5 r^5 - r^6) #1^3 + (r^4 + 4 r^5 + 3 r^6) #1^4 + (-r^5 - 3 r^6) #1^5 + r^6 #1^6 &, 5]], hp[
    Root[1 + r + r^2 + (-r - 2 r^2 - 2 r^3 - r^4) #1 + (r^2 + 3 r^3 + 3 r^4 + 2 r^5) #1^2 + (-r^3 - 3 r^4 - 5 r^5 - r^6)
        #1^3 + (r^4 + 4 r^5 + 3 r^6) #1^4 + (-r^5 - 3 r^6) #1^5 + r^6 #1^6 &, 6]]}, {r, 3.8, 4}]
```

Out[194]=




```
In[195]:= NSolve[Abs[hp[Root[1 + r + r^2 + (-r - 2 r^2 - 2 r^3 - r^4) #1 + (r^2 + 3 r^3 + 3 r^4 + 2 r^5) #1^2 +
        (-r^3 - 3 r^4 - 5 r^5 - r^6) #1^3 + (r^4 + 4 r^5 + 3 r^6) #1^4 + (-r^5 - 3 r^6) #1^5 + r^6 #1^6 &, 1]]] == 1]

Out[195]= {{r → -1.8415}, {r → -1.82843}, {r → 3.82843}, {r → 3.8415}}
```

From this we can see that the 3 period orbit lies approximately between r=3.82843 and r=3.8415

Four Period Orbits

```
In[196]:= Solve[k[x] == x, x]
```

Out[196]= $\left\{\{x \to 0\}, \left\{x \to \frac{-1+r}{r}\right\}, \left\{x \to \frac{r + r^2 - r\sqrt{-3 - 2r + r^2}}{2 r^2}\right\}, \left\{x \to \frac{r + r^2 + r\sqrt{-3 - 2r + r^2}}{2 r^2}\right\},\right.$

```
{x → Root[1 + r^2 + (-r^2 - r^3 - r^4 - r^5) #1 + (2 r^3 + r^4 + 4 r^5 + r^6 + 2 r^7) #1^2 +
     (-r^3 - 5 r^5 - 4 r^6 - 5 r^7 - 4 r^8 - r^9) #1^3 + (2 r^5 + 6 r^6 + 4 r^7 + 14 r^8 + 5 r^9 + 3 r^10) #1^4 +
     (-4 r^6 - r^7 - 18 r^8 - 12 r^9 - 12 r^10 - 3 r^11) #1^5 + (r^6 + 10 r^8 + 17 r^9 + 18 r^10 + 15 r^11 + r^12)
      #1^6 + (-2 r^8 - 14 r^9 - 12 r^10 - 30 r^11 - 6 r^12) #1^7 + (6 r^9 + 3 r^10 + 30 r^11 + 15 r^12) #1^8 +
     (-r^9 - 15 r^11 - 20 r^12) #1^9 + (3 r^11 + 15 r^12) #1^10 - 6 r^12 #1^11 + r^12 #1^12 &, 1]},
{x → Root[1 + r^2 + (-r^2 - r^3 - r^4 - r^5) #1 + (2 r^3 + r^4 + 4 r^5 + r^6 + 2 r^7) #1^2 +
     (-r^3 - 5 r^5 - 4 r^6 - 5 r^7 - 4 r^8 - r^9) #1^3 + (2 r^5 + 6 r^6 + 4 r^7 + 14 r^8 + 5 r^9 + 3 r^10) #1^4 +
     (-4 r^6 - r^7 - 18 r^8 - 12 r^9 - 12 r^10 - 3 r^11) #1^5 + (r^6 + 10 r^8 + 17 r^9 + 18 r^10 + 15 r^11 + r^12)
      #1^6 + (-2 r^8 - 14 r^9 - 12 r^10 - 30 r^11 - 6 r^12) #1^7 + (6 r^9 + 3 r^10 + 30 r^11 + 15 r^12) #1^8 +
     (-r^9 - 15 r^11 - 20 r^12) #1^9 + (3 r^11 + 15 r^12) #1^10 - 6 r^12 #1^11 + r^12 #1^12 &, 2]},
{x → Root[1 + r^2 + (-r^2 - r^3 - r^4 - r^5) #1 + (2 r^3 + r^4 + 4 r^5 + r^6 + 2 r^7) #1^2 +
```

$$(-r^3-5 r^5-4 r^6-5 r^7-4 r^8-r^9)\ \#1^3+(2 r^5+6 r^6+4 r^7+14 r^8+5 r^9+3 r^{10})\ \#1^4+$$
$$(-4 r^6-r^7-18 r^8-12 r^9-12 r^{10}-3 r^{11})\ \#1^5+(r^6+10 r^8+17 r^9+18 r^{10}+15 r^{11}+r^{12})$$
$$\#1^6+(-2 r^8-14 r^9-12 r^{10}-30 r^{11}-6 r^{12})\ \#1^7+(6 r^9+3 r^{10}+30 r^{11}+15 r^{12})\ \#1^8+$$
$$(-r^9-15 r^{11}-20 r^{12})\ \#1^9+(3 r^{11}+15 r^{12})\ \#1^{10}-6 r^{12}\ \#1^{11}+r^{12}\ \#1^{12}\ \&,\ 3]\},$$

$$\{x\to \mathbf{Root}[1+r^2+(-r^2-r^3-r^4-r^5)\ \#1+(2 r^3+r^4+4 r^5+r^6+2 r^7)\ \#1^2+$$
$$(-r^3-5 r^5-4 r^6-5 r^7-4 r^8-r^9)\ \#1^3+(2 r^5+6 r^6+4 r^7+14 r^8+5 r^9+3 r^{10})\ \#1^4+$$
$$(-4 r^6-r^7-18 r^8-12 r^9-12 r^{10}-3 r^{11})\ \#1^5+(r^6+10 r^8+17 r^9+18 r^{10}+15 r^{11}+r^{12})$$
$$\#1^6+(-2 r^8-14 r^9-12 r^{10}-30 r^{11}-6 r^{12})\ \#1^7+(6 r^9+3 r^{10}+30 r^{11}+15 r^{12})\ \#1^8+$$
$$(-r^9-15 r^{11}-20 r^{12})\ \#1^9+(3 r^{11}+15 r^{12})\ \#1^{10}-6 r^{12}\ \#1^{11}+r^{12}\ \#1^{12}\ \&,\ 4]\},$$

$$\{x\to \mathbf{Root}[1+r^2+(-r^2-r^3-r^4-r^5)\ \#1+(2 r^3+r^4+4 r^5+r^6+2 r^7)\ \#1^2+$$
$$(-r^3-5 r^5-4 r^6-5 r^7-4 r^8-r^9)\ \#1^3+(2 r^5+6 r^6+4 r^7+14 r^8+5 r^9+3 r^{10})\ \#1^4+$$
$$(-4 r^6-r^7-18 r^8-12 r^9-12 r^{10}-3 r^{11})\ \#1^5+(r^6+10 r^8+17 r^9+18 r^{10}+15 r^{11}+r^{12})$$
$$\#1^6+(-2 r^8-14 r^9-12 r^{10}-30 r^{11}-6 r^{12})\ \#1^7+(6 r^9+3 r^{10}+30 r^{11}+15 r^{12})\ \#1^8+$$
$$(-r^9-15 r^{11}-20 r^{12})\ \#1^9+(3 r^{11}+15 r^{12})\ \#1^{10}-6 r^{12}\ \#1^{11}+r^{12}\ \#1^{12}\ \&,\ 5]\},$$

$$\{x\to \mathbf{Root}[1+r^2+(-r^2-r^3-r^4-r^5)\ \#1+(2 r^3+r^4+4 r^5+r^6+2 r^7)\ \#1^2+$$
$$(-r^3-5 r^5-4 r^6-5 r^7-4 r^8-r^9)\ \#1^3+(2 r^5+6 r^6+4 r^7+14 r^8+5 r^9+3 r^{10})\ \#1^4+$$
$$(-4 r^6-r^7-18 r^8-12 r^9-12 r^{10}-3 r^{11})\ \#1^5+(r^6+10 r^8+17 r^9+18 r^{10}+15 r^{11}+r^{12})$$
$$\#1^6+(-2 r^8-14 r^9-12 r^{10}-30 r^{11}-6 r^{12})\ \#1^7+(6 r^9+3 r^{10}+30 r^{11}+15 r^{12})\ \#1^8+$$
$$(-r^9-15 r^{11}-20 r^{12})\ \#1^9+(3 r^{11}+15 r^{12})\ \#1^{10}-6 r^{12}\ \#1^{11}+r^{12}\ \#1^{12}\ \&,\ 6]\},$$

$$\{x\to \mathbf{Root}[1+r^2+(-r^2-r^3-r^4-r^5)\ \#1+(2 r^3+r^4+4 r^5+r^6+2 r^7)\ \#1^2+$$
$$(-r^3-5 r^5-4 r^6-5 r^7-4 r^8-r^9)\ \#1^3+(2 r^5+6 r^6+4 r^7+14 r^8+5 r^9+3 r^{10})\ \#1^4+$$
$$(-4 r^6-r^7-18 r^8-12 r^9-12 r^{10}-3 r^{11})\ \#1^5+(r^6+10 r^8+17 r^9+18 r^{10}+15 r^{11}+r^{12})$$
$$\#1^6+(-2 r^8-14 r^9-12 r^{10}-30 r^{11}-6 r^{12})\ \#1^7+(6 r^9+3 r^{10}+30 r^{11}+15 r^{12})\ \#1^8+$$
$$(-r^9-15 r^{11}-20 r^{12})\ \#1^9+(3 r^{11}+15 r^{12})\ \#1^{10}-6 r^{12}\ \#1^{11}+r^{12}\ \#1^{12}\ \&,\ 7]\},$$

$$\{x\to \mathbf{Root}[1+r^2+(-r^2-r^3-r^4-r^5)\ \#1+(2 r^3+r^4+4 r^5+r^6+2 r^7)\ \#1^2+$$
$$(-r^3-5 r^5-4 r^6-5 r^7-4 r^8-r^9)\ \#1^3+(2 r^5+6 r^6+4 r^7+14 r^8+5 r^9+3 r^{10})\ \#1^4+$$
$$(-4 r^6-r^7-18 r^8-12 r^9-12 r^{10}-3 r^{11})\ \#1^5+(r^6+10 r^8+17 r^9+18 r^{10}+15 r^{11}+r^{12})$$
$$\#1^6+(-2 r^8-14 r^9-12 r^{10}-30 r^{11}-6 r^{12})\ \#1^7+(6 r^9+3 r^{10}+30 r^{11}+15 r^{12})\ \#1^8+$$
$$(-r^9-15 r^{11}-20 r^{12})\ \#1^9+(3 r^{11}+15 r^{12})\ \#1^{10}-6 r^{12}\ \#1^{11}+r^{12}\ \#1^{12}\ \&,\ 8]\},$$

$$\{x\to \mathbf{Root}[1+r^2+(-r^2-r^3-r^4-r^5)\ \#1+(2 r^3+r^4+4 r^5+r^6+2 r^7)\ \#1^2+$$
$$(-r^3-5 r^5-4 r^6-5 r^7-4 r^8-r^9)\ \#1^3+(2 r^5+6 r^6+4 r^7+14 r^8+5 r^9+3 r^{10})\ \#1^4+$$
$$(-4 r^6-r^7-18 r^8-12 r^9-12 r^{10}-3 r^{11})\ \#1^5+(r^6+10 r^8+17 r^9+18 r^{10}+15 r^{11}+r^{12})$$
$$\#1^6+(-2 r^8-14 r^9-12 r^{10}-30 r^{11}-6 r^{12})\ \#1^7+(6 r^9+3 r^{10}+30 r^{11}+15 r^{12})\ \#1^8+$$
$$(-r^9-15 r^{11}-20 r^{12})\ \#1^9+(3 r^{11}+15 r^{12})\ \#1^{10}-6 r^{12}\ \#1^{11}+r^{12}\ \#1^{12}\ \&,\ 9]\},$$

$$\{x\to \mathbf{Root}[1+r^2+(-r^2-r^3-r^4-r^5)\ \#1+(2 r^3+r^4+4 r^5+r^6+2 r^7)\ \#1^2+$$
$$(-r^3-5 r^5-4 r^6-5 r^7-4 r^8-r^9)\ \#1^3+(2 r^5+6 r^6+4 r^7+14 r^8+5 r^9+3 r^{10})\ \#1^4+$$
$$(-4 r^6-r^7-18 r^8-12 r^9-12 r^{10}-3 r^{11})\ \#1^5+(r^6+10 r^8+17 r^9+18 r^{10}+15 r^{11}+r^{12})$$
$$\#1^6+(-2 r^8-14 r^9-12 r^{10}-30 r^{11}-6 r^{12})\ \#1^7+(6 r^9+3 r^{10}+30 r^{11}+15 r^{12})\ \#1^8+$$
$$(-r^9-15 r^{11}-20 r^{12})\ \#1^9+(3 r^{11}+15 r^{12})\ \#1^{10}-6 r^{12}\ \#1^{11}+r^{12}\ \#1^{12}\ \&,\ 10]\},$$

$$\{x\to \mathbf{Root}[1+r^2+(-r^2-r^3-r^4-r^5)\ \#1+(2 r^3+r^4+4 r^5+r^6+2 r^7)\ \#1^2+$$
$$(-r^3-5 r^5-4 r^6-5 r^7-4 r^8-r^9)\ \#1^3+(2 r^5+6 r^6+4 r^7+14 r^8+5 r^9+3 r^{10})\ \#1^4+$$
$$(-4 r^6-r^7-18 r^8-12 r^9-12 r^{10}-3 r^{11})\ \#1^5+(r^6+10 r^8+17 r^9+18 r^{10}+15 r^{11}+r^{12})$$
$$\#1^6+(-2 r^8-14 r^9-12 r^{10}-30 r^{11}-6 r^{12})\ \#1^7+(6 r^9+3 r^{10}+30 r^{11}+15 r^{12})\ \#1^8+$$
$$(-r^9-15 r^{11}-20 r^{12})\ \#1^9+(3 r^{11}+15 r^{12})\ \#1^{10}-6 r^{12}\ \#1^{11}+r^{12}\ \#1^{12}\ \&,\ 11]\},$$

$$\{x\to \mathbf{Root}[1+r^2+(-r^2-r^3-r^4-r^5)\ \#1+(2 r^3+r^4+4 r^5+r^6+2 r^7)\ \#1^2+$$
$$(-r^3-5 r^5-4 r^6-5 r^7-4 r^8-r^9)\ \#1^3+(2 r^5+6 r^6+4 r^7+14 r^8+5 r^9+3 r^{10})\ \#1^4+$$
$$(-4 r^6-r^7-18 r^8-12 r^9-12 r^{10}-3 r^{11})\ \#1^5+(r^6+10 r^8+17 r^9+18 r^{10}+15 r^{11}+r^{12})$$
$$\#1^6+(-2 r^8-14 r^9-12 r^{10}-30 r^{11}-6 r^{12})\ \#1^7+(6 r^9+3 r^{10}+30 r^{11}+15 r^{12})\ \#1^8+$$

```
In[197]:= FullSimplify[D[k[x], x]]

Out[197]= -r^4 (-1 + 2 x) (1 + 2 r (-1 + x) x) (1 + 2 r^2 (-1 + x) x (1 + r (-1 + x) x))
           (1 + 2 r^3 (-1 + x) x (1 + r (-1 + x) x) (1 + r^2 (-1 + x) x (1 + r (-1 + x) x)))

In[198]:= kp[x_] := -r^4 (-1 + 2 x) (1 + 2 r (-1 + x) x) (1 + 2 r^2 (-1 + x) x (1 + r (-1 + x) x))
              (1 + 2 r^3 (-1 + x) x (1 + r (-1 + x) x) (1 + r^2 (-1 + x) x (1 + r (-1 + x) x)));

In[210]:= Plot[{kp[Root[1 + r^2 + (-r^2 - r^3 - r^4 - r^5) #1 + (2 r^3 + r^4 + 4 r^5 + r^6 + 2 r^7) #1^2 +
              (-r^3 - 5 r^5 - 4 r^6 - 5 r^7 - 4 r^8 - r^9) #1^3 + (2 r^5 + 6 r^6 + 4 r^7 + 14 r^8 + 5 r^9 + 3 r^10) #1^4 +
              (-4 r^6 - r^7 - 18 r^8 - 12 r^9 - 12 r^10 - 3 r^11) #1^5 + (r^6 + 10 r^8 + 17 r^9 + 18 r^10 + 15 r^11 + r^12)
               #1^6 + (-2 r^8 - 14 r^9 - 12 r^10 - 30 r^11 - 6 r^12) #1^7 + (6 r^9 + 3 r^10 + 30 r^11 + 15 r^12) #1^8 +
              (-r^9 - 15 r^11 - 20 r^12) #1^9 + (3 r^11 + 15 r^12) #1^10 - 6 r^12 #1^11 + r^12 #1^12 &, 1]],
    kp[Root[1 + r^2 + (-r^2 - r^3 - r^4 - r^5) #1 + (2 r^3 + r^4 + 4 r^5 + r^6 + 2 r^7) #1^2 +
              (-r^3 - 5 r^5 - 4 r^6 - 5 r^7 - 4 r^8 - r^9) #1^3 + (2 r^5 + 6 r^6 + 4 r^7 + 14 r^8 + 5 r^9 + 3 r^10) #1^4 +
              (-4 r^6 - r^7 - 18 r^8 - 12 r^9 - 12 r^10 - 3 r^11) #1^5 + (r^6 + 10 r^8 + 17 r^9 + 18 r^10 + 15 r^11 + r^12)
               #1^6 + (-2 r^8 - 14 r^9 - 12 r^10 - 30 r^11 - 6 r^12) #1^7 + (6 r^9 + 3 r^10 + 30 r^11 + 15 r^12) #1^8 +
              (-r^9 - 15 r^11 - 20 r^12) #1^9 + (3 r^11 + 15 r^12) #1^10 - 6 r^12 #1^11 + r^12 #1^12 &, 2]],
    kp[Root[1 + r^2 + (-r^2 - r^3 - r^4 - r^5) #1 + (2 r^3 + r^4 + 4 r^5 + r^6 + 2 r^7) #1^2 +
              (-r^3 - 5 r^5 - 4 r^6 - 5 r^7 - 4 r^8 - r^9) #1^3 + (2 r^5 + 6 r^6 + 4 r^7 + 14 r^8 + 5 r^9 + 3 r^10) #1^4 +
              (-4 r^6 - r^7 - 18 r^8 - 12 r^9 - 12 r^10 - 3 r^11) #1^5 + (r^6 + 10 r^8 + 17 r^9 + 18 r^10 + 15 r^11 + r^12)
               #1^6 + (-2 r^8 - 14 r^9 - 12 r^10 - 30 r^11 - 6 r^12) #1^7 + (6 r^9 + 3 r^10 + 30 r^11 + 15 r^12) #1^8 +
              (-r^9 - 15 r^11 - 20 r^12) #1^9 + (3 r^11 + 15 r^12) #1^10 - 6 r^12 #1^11 + r^12 #1^12 &, 3]],
    kp[Root[1 + r^2 + (-r^2 - r^3 - r^4 - r^5) #1 + (2 r^3 + r^4 + 4 r^5 + r^6 + 2 r^7) #1^2 +
              (-r^3 - 5 r^5 - 4 r^6 - 5 r^7 - 4 r^8 - r^9) #1^3 + (2 r^5 + 6 r^6 + 4 r^7 + 14 r^8 + 5 r^9 + 3 r^10) #1^4 +
              (-4 r^6 - r^7 - 18 r^8 - 12 r^9 - 12 r^10 - 3 r^11) #1^5 + (r^6 + 10 r^8 + 17 r^9 + 18 r^10 + 15 r^11 + r^12)
               #1^6 + (-2 r^8 - 14 r^9 - 12 r^10 - 30 r^11 - 6 r^12) #1^7 + (6 r^9 + 3 r^10 + 30 r^11 + 15 r^12) #1^8 +
              (-r^9 - 15 r^11 - 20 r^12) #1^9 + (3 r^11 + 15 r^12) #1^10 - 6 r^12 #1^11 + r^12 #1^12 &, 4]],
    kp[Root[1 + r^2 + (-r^2 - r^3 - r^4 - r^5) #1 + (2 r^3 + r^4 + 4 r^5 + r^6 + 2 r^7) #1^2 +
              (-r^3 - 5 r^5 - 4 r^6 - 5 r^7 - 4 r^8 - r^9) #1^3 + (2 r^5 + 6 r^6 + 4 r^7 + 14 r^8 + 5 r^9 + 3 r^10) #1^4 +
              (-4 r^6 - r^7 - 18 r^8 - 12 r^9 - 12 r^10 - 3 r^11) #1^5 + (r^6 + 10 r^8 + 17 r^9 + 18 r^10 + 15 r^11 + r^12)
               #1^6 + (-2 r^8 - 14 r^9 - 12 r^10 - 30 r^11 - 6 r^12) #1^7 + (6 r^9 + 3 r^10 + 30 r^11 + 15 r^12) #1^8 +
              (-r^9 - 15 r^11 - 20 r^12) #1^9 + (3 r^11 + 15 r^12) #1^10 - 6 r^12 #1^11 + r^12 #1^12 &, 5]],
    kp[Root[1 + r^2 + (-r^2 - r^3 - r^4 - r^5) #1 + (2 r^3 + r^4 + 4 r^5 + r^6 + 2 r^7) #1^2 +
              (-r^3 - 5 r^5 - 4 r^6 - 5 r^7 - 4 r^8 - r^9) #1^3 + (2 r^5 + 6 r^6 + 4 r^7 + 14 r^8 + 5 r^9 + 3 r^10) #1^4 +
              (-4 r^6 - r^7 - 18 r^8 - 12 r^9 - 12 r^10 - 3 r^11) #1^5 + (r^6 + 10 r^8 + 17 r^9 + 18 r^10 + 15 r^11 + r^12)
               #1^6 + (-2 r^8 - 14 r^9 - 12 r^10 - 30 r^11 - 6 r^12) #1^7 + (6 r^9 + 3 r^10 + 30 r^11 + 15 r^12) #1^8 +
              (-r^9 - 15 r^11 - 20 r^12) #1^9 + (3 r^11 + 15 r^12) #1^10 - 6 r^12 #1^11 + r^12 #1^12 &, 6]],
    kp[Root[1 + r^2 + (-r^2 - r^3 - r^4 - r^5) #1 + (2 r^3 + r^4 + 4 r^5 + r^6 + 2 r^7) #1^2 +
              (-r^3 - 5 r^5 - 4 r^6 - 5 r^7 - 4 r^8 - r^9) #1^3 + (2 r^5 + 6 r^6 + 4 r^7 + 14 r^8 + 5 r^9 + 3 r^10) #1^4 +
              (-4 r^6 - r^7 - 18 r^8 - 12 r^9 - 12 r^10 - 3 r^11) #1^5 + (r^6 + 10 r^8 + 17 r^9 + 18 r^10 + 15 r^11 + r^12)
               #1^6 + (-2 r^8 - 14 r^9 - 12 r^10 - 30 r^11 - 6 r^12) #1^7 + (6 r^9 + 3 r^10 + 30 r^11 + 15 r^12) #1^8 +
              (-r^9 - 15 r^11 - 20 r^12) #1^9 + (3 r^11 + 15 r^12) #1^10 - 6 r^12 #1^11 + r^12 #1^12 &, 7]],
    kp[Root[1 + r^2 + (-r^2 - r^3 - r^4 - r^5) #1 + (2 r^3 + r^4 + 4 r^5 + r^6 + 2 r^7) #1^2 +
              (-r^3 - 5 r^5 - 4 r^6 - 5 r^7 - 4 r^8 - r^9) #1^3 + (2 r^5 + 6 r^6 + 4 r^7 + 14 r^8 + 5 r^9 + 3 r^10) #1^4 +
              (-4 r^6 - r^7 - 18 r^8 - 12 r^9 - 12 r^10 - 3 r^11) #1^5 + (r^6 + 10 r^8 + 17 r^9 + 18 r^10 + 15 r^11 + r^12)
               #1^6 + (-2 r^8 - 14 r^9 - 12 r^10 - 30 r^11 - 6 r^12) #1^7 + (6 r^9 + 3 r^10 + 30 r^11 + 15 r^12) #1^8 +
              (-r^9 - 15 r^11 - 20 r^12) #1^9 + (3 r^11 + 15 r^12) #1^10 - 6 r^12 #1^11 + r^12 #1^12 &, 1]],
    kp[Root[1 + r^2 + (-r^2 - r^3 - r^4 - r^5) #1 + (2 r^3 + r^4 + 4 r^5 + r^6 + 2 r^7) #1^2 +
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    (-r^3 - 5 r^5 - 4 r^6 - 5 r^7 - 4 r^8 - r^9) #1^3 + (2 r^5 + 6 r^6 + 4 r^7 + 14 r^8 + 5 r^9 + 3 r^10) #1^4 +
    (-4 r^6 - r^7 - 18 r^8 - 12 r^9 - 12 r^10 - 3 r^11) #1^5 + (r^6 + 10 r^8 + 17 r^9 + 18 r^10 + 15 r^11 + r^12)
     #1^6 + (-2 r^8 - 14 r^9 - 12 r^10 - 30 r^11 - 6 r^12) #1^7 + (6 r^9 + 3 r^10 + 30 r^11 + 15 r^12) #1^8 +
    (-r^9 - 15 r^11 - 20 r^12) #1^9 + (3 r^11 + 15 r^12) #1^10 - 6 r^12 #1^11 + r^12 #1^12 &, 8]],
kp[Root[1 + r^2 + (-r^2 - r^3 - r^4 - r^5) #1 + (2 r^3 + r^4 + 4 r^5 + r^6 + 2 r^7) #1^2 +
    (-r^3 - 5 r^5 - 4 r^6 - 5 r^7 - 4 r^8 - r^9) #1^3 + (2 r^5 + 6 r^6 + 4 r^7 + 14 r^8 + 5 r^9 + 3 r^10) #1^4 +
    (-4 r^6 - r^7 - 18 r^8 - 12 r^9 - 12 r^10 - 3 r^11) #1^5 + (r^6 + 10 r^8 + 17 r^9 + 18 r^10 + 15 r^11 + r^12)
     #1^6 + (-2 r^8 - 14 r^9 - 12 r^10 - 30 r^11 - 6 r^12) #1^7 + (6 r^9 + 3 r^10 + 30 r^11 + 15 r^12) #1^8 +
    (-r^9 - 15 r^11 - 20 r^12) #1^9 + (3 r^11 + 15 r^12) #1^10 - 6 r^12 #1^11 + r^12 #1^12 &, 9]],
kp[Root[1 + r^2 + (-r^2 - r^3 - r^4 - r^5) #1 + (2 r^3 + r^4 + 4 r^5 + r^6 + 2 r^7) #1^2 +
    (-r^3 - 5 r^5 - 4 r^6 - 5 r^7 - 4 r^8 - r^9) #1^3 + (2 r^5 + 6 r^6 + 4 r^7 + 14 r^8 + 5 r^9 + 3 r^10) #1^4 +
    (-4 r^6 - r^7 - 18 r^8 - 12 r^9 - 12 r^10 - 3 r^11) #1^5 + (r^6 + 10 r^8 + 17 r^9 + 18 r^10 + 15 r^11 + r^12)
     #1^6 + (-2 r^8 - 14 r^9 - 12 r^10 - 30 r^11 - 6 r^12) #1^7 + (6 r^9 + 3 r^10 + 30 r^11 + 15 r^12) #1^8 +
    (-r^9 - 15 r^11 - 20 r^12) #1^9 + (3 r^11 + 15 r^12) #1^10 - 6 r^12 #1^11 + r^12 #1^12 &, 10]],
kp[Root[1 + r^2 + (-r^2 - r^3 - r^4 - r^5) #1 + (2 r^3 + r^4 + 4 r^5 + r^6 + 2 r^7) #1^2 +
    (-r^3 - 5 r^5 - 4 r^6 - 5 r^7 - 4 r^8 - r^9) #1^3 + (2 r^5 + 6 r^6 + 4 r^7 + 14 r^8 + 5 r^9 + 3 r^10) #1^4 +
    (-4 r^6 - r^7 - 18 r^8 - 12 r^9 - 12 r^10 - 3 r^11) #1^5 + (r^6 + 10 r^8 + 17 r^9 + 18 r^10 + 15 r^11 + r^12)
     #1^6 + (-2 r^8 - 14 r^9 - 12 r^10 - 30 r^11 - 6 r^12) #1^7 + (6 r^9 + 3 r^10 + 30 r^11 + 15 r^12) #1^8 +
    (-r^9 - 15 r^11 - 20 r^12) #1^9 + (3 r^11 + 15 r^12) #1^10 - 6 r^12 #1^11 + r^12 #1^12 &, 11]],
kp[Root[1 + r^2 + (-r^2 - r^3 - r^4 - r^5) #1 + (2 r^3 + r^4 + 4 r^5 + r^6 + 2 r^7) #1^2 +
    (-r^3 - 5 r^5 - 4 r^6 - 5 r^7 - 4 r^8 - r^9) #1^3 + (2 r^5 + 6 r^6 + 4 r^7 + 14 r^8 + 5 r^9 + 3 r^10) #1^4 +
    (-4 r^6 - r^7 - 18 r^8 - 12 r^9 - 12 r^10 - 3 r^11) #1^5 + (r^6 + 10 r^8 + 17 r^9 + 18 r^10 + 15 r^11 + r^12)
     #1^6 + (-2 r^8 - 14 r^9 - 12 r^10 - 30 r^11 - 6 r^12) #1^7 + (6 r^9 + 3 r^10 + 30 r^11 + 15 r^12) #1^8 +
    (-r^9 - 15 r^11 - 20 r^12) #1^9 + (3 r^11 + 15 r^12) #1^10 - 6 r^12 #1^11 + r^12 #1^12 &, 12]],
1, -1}, {r, 3.9601, 3.9608}, PlotPoints → 200, PlotRange →
{-2, 2}]
```
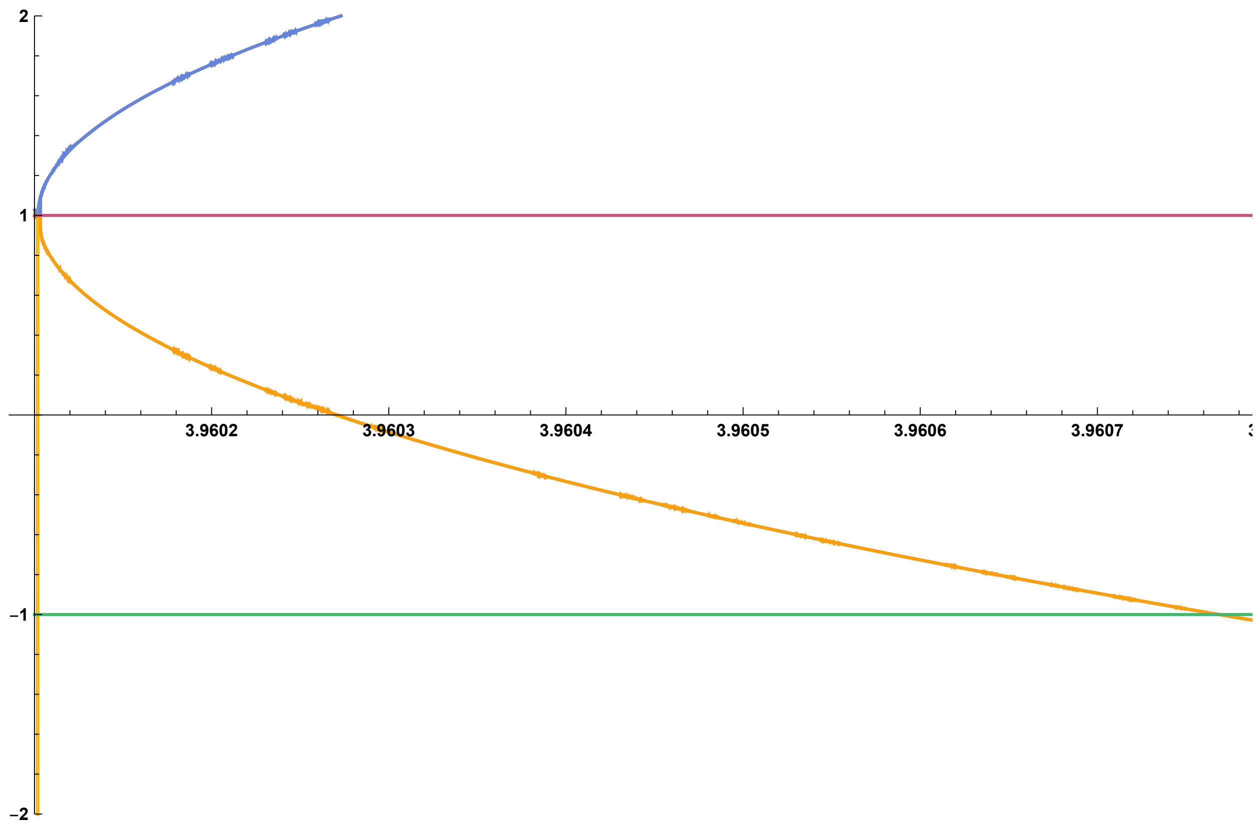

Out[210]=



From this we can see that the 4 period orbit lies approximately between r=3.9601 and r=3.96075

#3.

Consider the discrete dynamical system

$$X_{n+1} = aX_n e^{-X_n} \text{ and } x_0 > 0,$$

where $a > 0$.

a.) Find the fixed points and analyze their stability.

Solution:

Fixed points satisfy:

$$x^* = a x^* e^{-x^*}$$

$$\Rightarrow x^* = 0 \text{ or } a = e^{x^*}$$

$$\Rightarrow x^* = 0 \text{ or } x^* = \ln(a)$$

Furthermore, letting $f(x) = axe^{-x}$ it follows that

$$f'(x) = ae^{-x}(1-x).$$

$$\Rightarrow f'(0) = a \text{ and } f'(\ln(a)) = 1 - \ln(a)$$

Consequently, 0 is stable if $0 \leq a < 1$ while if $1 < a < e^2$ it follows that $\ln(a)$ is stable.

b.) Show that for all n, $x_n \geq 0$.

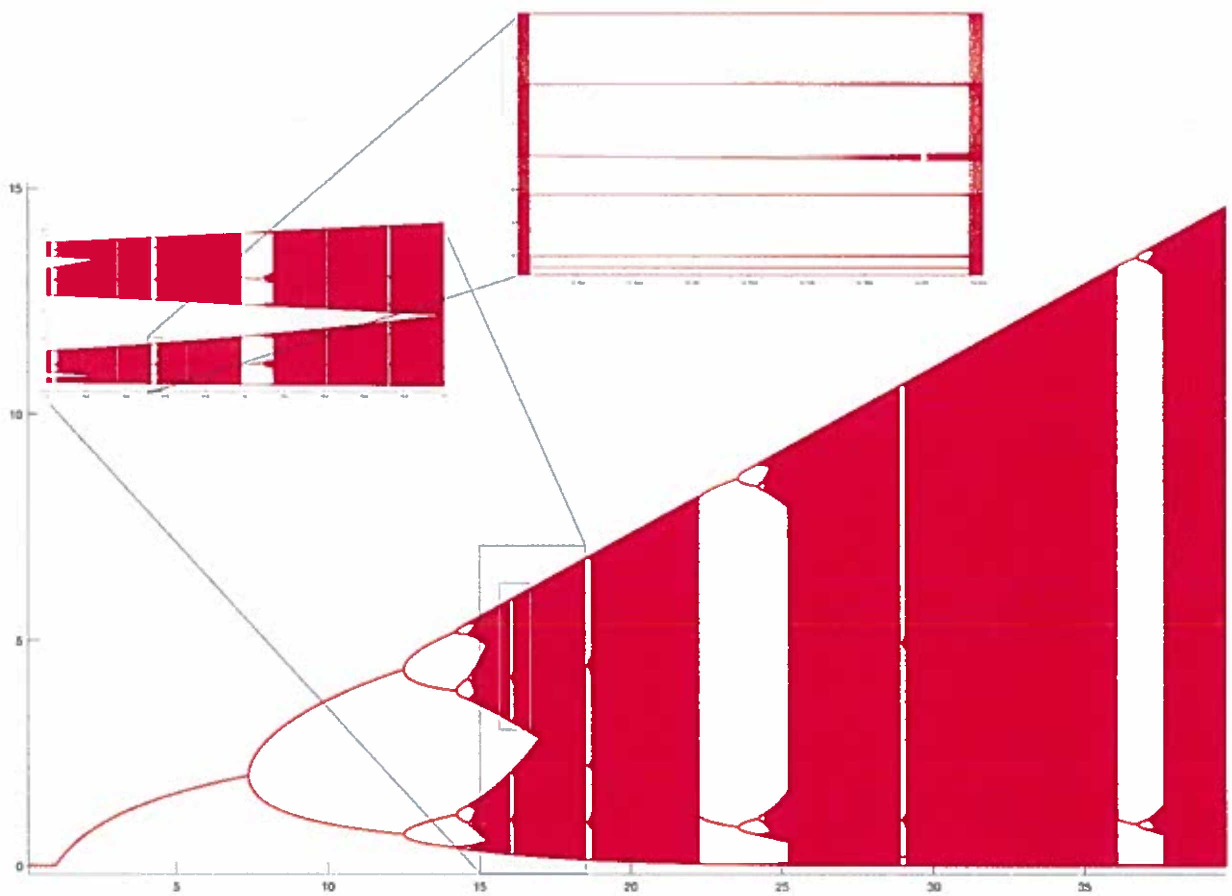
Solution:

Since $f > 0$ and $x_0 > 0$ it follows that $x_{n+1} = f(x_n) > 0$.

C. Create an orbital diagram

Solution:

See attached.



15
10
5
0
5
10
15
20
25
30
35