Homework 9

Mathematical Modeling

Due: December 3, 2018

1 Problems for Everybody

1. Consider the Lorenz equations:
$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = rx - y - xz \\ \dot{z} = xy - bz \end{cases}$$
where $\sigma > 0$, $r > 0$, and $b > 0$ are constants.
   (a) Show that the fixed points for this system are $(0, 0, 0)$ and $C_{\pm} = \left(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1\right)$. For what values of r will $C_{\pm}$ exist. **Hint:** To show something is true you can simply substitute into the equations.
   (b) Analyze the local stability of the fixed point $(0, 0, 0)$. What type of bifurcation occurs in r?
   (c) Show that for $r < 1$ the origin $(0, 0, 0)$ is globally stable by considering the following Lyapunov function: $V(x, y, z) = \sigma^{-1}x^2 + y^2 + z^2$.
   (d) Show that the characteristic equations for the eigenvalues of the Jacobian matrix at $C_{\pm}$ is
   $$\lambda^3 + (\sigma + b + 1)\lambda^2 + (r + \sigma)b\lambda + 2b\sigma(r - 1) = 0.$$
   By seeking solutions of the form $\lambda = i\omega$ show that there is a pair of pure imaginary eigenvalues when
   $$r_H = \sigma\left(\frac{\sigma + b + 3}{\sigma - b - b}\right).$$
   Explain why we need to assume $\sigma > b + 1$.
   (e) Show that there is a certain ellipsoidal region E of the form $rx^2 + \sigma y^2 + \sigma(z - 2r)^2 \leq C$ such that all trajectories of the Lorenz equations eventually enter E and stay in there forever.
   (f) Show that the z-axis is an invariant line for the Lorenz equations.

2. In the course we have been using the concept of an attractor without properly defining it. An **attractor** is a closed set A satisfying the following properties:
   - *A is an invariant set*: any trajectory $\mathbf{x}(t)$ that starts in A stays in A for all time.
   - *A attracts on open set of initial conditions:* there is an open set U containing A such that if $\mathbf{x}(0) \in U$, then the distance from $\mathbf{x}(t)$ to A tends to zero as $t \to \infty$. This means that A attracts all trajectories that start sufficiently close to it. That largest such U is called the *basin of attraction.*
   - *A is minimal*: there is no proper subset of A that satisfies the above two conditions.

   Now consider the following system in polar coordinates:
   $$\begin{cases} \dot{r} = r(1 - r^2)\dot{\theta} = 1 \end{cases}.$$
   Let D be the disk $x^2 + y^2 \leq 1$.

(a) Is D an invariant set?

(b) Does D attract an open set of initial conditions?

(c) Is D an attractor? If not, why not? If so, find its basin of attraction.

(d) Repeat part (c) for the circle $x^2 + y^2 = 1$.